VALUES AND PRICES IN A GENERAL INPUT-OUTPUT MODEL

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ABSTRACT

This paper is to define values and prices in a general linear model, by showing how to conceive and calculate direct and indirect requirements of a standard commodity or labour to produce one unit of various commodities in an extended von Neumann-Morishima model of joint production. This is a simple application of a principle of algebra, and so is a straightforward generalization of the concept of labour value well known for a simple Leontief model of circulating capital. In our model, proper joint production, heterogeneous labour, durable consumption goods, and disposal processes for “bads” are all allowed for. We also extend the scope of values and show how to use our definition of values to define unskilled labour, and suggest how to distinguish between necessities and luxuries. International trade is also introduced to our model. We employ many numerical examples to help the reader to grasp our method in a concrete way.

1. INTRODUCTION

This paper presents an extended version of von Neumann-Morishima models of joint production in the spirit of classical economists. In the original model by von Neumann (1945-46), labour inputs are implicit as they are included in the

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1 This paper was written while the first author was a visiting professor at Department of Economic Sciences, University of Padova. He is grateful to the Department for its hospitality. Thanks are due to Professor Sergio Parrinello for his comments.

2 Kurz (1986) explains how classical economists as well as early neoclassical economists tackled the problems posed by joint production, and concluded: “These considerations make it clear how important is the development of a theory of output which will be in harmony with the classical theory of value and distribution”. (Kurz (1986), p.35).
input coefficient matrix together with material inputs. Likewise, some 20 years before, P. Sraffa formulated an elementary input-output model in which, to use J. Mill’s phrase, “the agents of production [were] the commodities themselves” (see Kurz (2006), p.103), and which have been probably inspired by Petty’s notion of “physical real cost”. Morishima (1973, 1974) took out the labour input vector, and the consumption basket for workers in an explicit way, enabling him to discuss about labour values as well as exploitation. The essence of our method is the complete symmetry in dealing with goods and labour, and we allow for general joint production, “bads”, and heterogeneous labour. To give a wider scope for values, we also discuss how to find out that a type of labour is less skilled than others, and how to discriminate luxuries from necessities without depending prices and/or demand theory. To be realistic, international trade is introduced toward the end of the paper.

A sharp distinction shall be made between values and prices. By the value of commodity \(i\) (or labour of type \(s\)) in terms of commodity \(j\) (or labour of type \(s\)), we loosely mean the amount of the latter that must be directly or indirectly “consumed” to make possible the production of one unit of the former as a net product. Our precise definition is given in section 4. By the price of a commodity (or a type of labour), we mean the long-period, competitive price of that commodity (or that type of labour) in terms of a chose numeraire. Values and prices are determined solely based on technical data in production as well as in household activities.

Our method is derived from Fujimoto and Fujita (2006) where labour is treated in the same manner to the other normal commodities in a symmetric way. (See also Jeong (1982, 1984).) It is an approach also based on inequality systems instead of equalities as started by von Neumann (1945-46) and recommended by Morishima (1964, 1974). Certainly as a linear model, we assume constant returns to scale throughout the paper. In section 2, we describe our method, using a Leontief model so that the reader can grasp more easily what we are going to advocate. Section 3 deals with a special model of joint production in which the square net output matrix is inverse-positive. Then we explain our model and method fully in section 4. Our model admits of joint production, heterogeneous labour, disposal processes and as many alternative processes as one likes. While up to section 4 values are discussed, the sections after 7 prices are the objects of investigation, with two interludes on bads and disposal processes in section 5, and necessities and luxuries in section 6. Section 7 explains how to determine “natural” prices in our general model of joint production. In section 8, land rent is explained, and section 9 takes up international trade. The final section 10 gives our conclusions.

2. VALUES IN A SIMPLE LEONTIEF MODEL

Let us start with a simple Leontief model of circulating capital. We consider how to calculate labour values for this model in our own way. We assume there
are $n$ kinds of commodities (goods and services with no “bads”) and one type of homogeneous labour. Let $A$ be the $n \times n$ material (and service) input coefficient matrix, $I$ the $n \times n$ or $(n+1) \times (n+1)$ identity matrix depending on the context. The symbol, $\ell$, means the row $n$-vector of labour input coefficients, and $c$ the column $n$-vector of workers’ consumption basket which enables the household activity to reproduce one unit of labour force in a production period. The row $n$-vector of labour values are written as $\lambda$. The symbol $R^n_+$ means the nonnegative orthant of the $n$-dimensional Euclidean space, $R^n$. A prime to a vector indicates its transposition. A subscript attached to a vector, as in $x_i$ or $(Ax)_i$, means the $i$-th element of the vector.

First, we make the following traditional assumption in Leontief models.

**Assumption (A0) (Productiveness assumption):** There exists a nonnegative column $n$-vector $x$ such that $x \geq Ax$.

Then as is well known now, the labour values are determined by

$$\lambda = \lambda A + \ell,$$

which has a nonnegative nonzero solution because of the above assumption, equivalent to the Hawkins-Simon condition (Hawkins and Simon (1949), Nikaido (1963)). The magnitude $\lambda_i$, the $i$-th entry of $\lambda$, stands for the amount of labour which has entered the $i$-th commodity directly or indirectly. It is the principle of algebra that if we know the amount of labour which has entered the $i$-th commodity directly or indirectly, and denote this by $\lambda_i$, then we obtain eq.(1). Eq.(1) can also be expressed as

$$\lambda = (I + A + A^2 + \ldots)\ell,$$

thus showing the successive dated labour inputs in terms of Carl Neumann series. This way of interpretation of direct and indirect labour input cannot be extended to the case of general joint production as Sraffa observed (Sraffa (1960), p.56 and p.58). So, it is better to forget about this expression.

The labour value of one unit of labour (force), $\lambda \ell$, is calculated as the inner product

$$\lambda \ell = \lambda c.$$

The reader may notice that the two equations (1) and (2) can be combined to lead to

$$\begin{pmatrix} \lambda & \lambda \ell \end{pmatrix} = \begin{pmatrix} \lambda & 1 \end{pmatrix} A,$$

where $(n+1) \times (n+1)$ matrix $A$ is defined as

$$A \equiv \begin{pmatrix} A & c \\ \ell & 0 \end{pmatrix}.$$  

This matrix is called the ‘complete matrix’ in Bródy(1970, p.23). It is important to note here that in eq.(3), the direct input of labour is given unity as its value.
on the RHS, while the value of one unit of labour is calculated as $\lambda_\ell$ on the LHS.

For the whole economy to be reproduced, or more precisely, expanded, the productiveness assumption above is not enough. We also need $\lambda_{n+1} \equiv \lambda_\ell < 1$. To incorporate this condition, we had better make

**Assumption (AL) (Productiveness assumption):** There exists a nonnegative column $(n+1)$-vector $x$ such that $x \succ A x$. Thus, $(I - A)^{-1} > 0$.

Eq.(3) is transformed as

$$\begin{pmatrix} \lambda & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \end{pmatrix} A + (0, \ldots, 0, 1 - \lambda_\ell).$$

Therefore,

$$\begin{pmatrix} \lambda & 1 \end{pmatrix} = (0, \ldots, 0, 1 - \lambda_\ell)(I - A)^{-1}.$$  

By denoting by $b_{ij}$ the $(i, j)$-element of the inverse matrix $(I - A)^{-1}$, we have

$$1 = b_{n+1,n+1}(1 - \lambda_\ell), \quad \text{that is,} \quad \lambda_\ell = \frac{b_{n+1,n+1} - 1}{b_{n+1,n+1}}, \quad \text{and}$$

$$\lambda_j = \frac{b_{n+1,j}}{b_{n+1,n+1}} \quad \text{for} \quad j = 1, \ldots, n.$$  

It naturally follows that $0 \leq \lambda_\ell < 1$ and $\lambda_j \geq 0$ for $j = 1, \ldots, n$, i.e., the labour value of one unit of labour is less than one. When labour is indispensable to produce a basket $c$, we have $0 < \lambda_\ell$. (See Fujimoto and Fujita (2006)).

Now, let us consider a linear programming problem like this:

$$(\text{PL}) \quad \text{min} \; x_{n+1} \; \text{subject to} \; x \geq Ax + c_{[n+1]} \; \text{and} \; x \in \mathbb{R}^{n+1},$$

where $c_{[n+1]} \equiv (0, \ldots, 0, 1)'$, i.e., the $(n+1)$-column vector whose $(n+1)$-th entry is unity with all the remaining elements being zero. This programming problem dictates that while producing one unit of labour force as net product, we should minimize the gross production of labour force. It is easy to notice that the objective function can also be $(Ax)_{n+1}$, i.e., the total input of labour. Thanks to our assumption, the inequality constraints in the problem (LM) above are in fact equalities in a unique optimal solution, and we know the optimum value of $x_{n+1}^*$ is $b_{n+1,n+1}$. In interpreting the problem (PL), again we can utilize the principle of algebra. If we put the optimum value of the problem $x_{n+1}^*$, and write the labour amount which has entered directly and indirectly into one unit of labour as $\lambda_\ell$, we have

$$1 = x_{n+1}^*(1 - \lambda_\ell).$$

Hence

$$\lambda_\ell = \frac{x_{n+1}^* - 1}{x_{n+1}^*}.$$
Since \( x_{n+1}^* = b_{n+1,n+1} \), we get the exactly same magnitude as in eq.(5) through a linear programming problem. It should be remembered that \((1 - \lambda)\) is a unitless pure number as a ratio, expressing the maximum own rate of surplus for labour. One can certainly regard \((1 - \lambda)\) or \(\lambda\) as carrying the same unit for labour within one unit of labour reproduced.

Next we have to turn to the task of determining labour values of other normal commodities. As the reader may easily expect, we consider the following linear programming problem dual to the above (LP).

\[(DL) \quad \text{max} \ q_{n+1} \text{ subject to } q' \leq q'A + e'_{[n+1]} \text{ and } q \in R_{+}^{n+1}.\]

Because of our assumption A1, the optimum vector \(q^* = (b_{n+1,1}, \ldots, b_{n+1,n}, b_{n+1,n+1})\).

Thus, the knowledge of eq.(6) gives us

\[\lambda_j = \frac{q_{n+1,j}^*}{q_{n+1,n+1}^*} \text{ for } j = 1, \ldots, n.\]

Encouraged by this, we may interpret the problem (DL) as follows. The very constraint of (DL) can be rewritten through dividing all the terms by \(q_{n+1}^*\), supposing \(q_{n+1}^* \neq 0\):

\[(q_1/q_{n+1}^*, \ldots, q_n/q_{n+1}^*, 1) \leq (q_1/q_{n+1}^*, \ldots, q_n/q_{n+1}^*, 1)A + (0, \ldots, 0, 1/q_{n+1}^*).\]

(8)

At the optimum, we know from eq.(7) and the duality that

\[1 = q_{n+1}^*(1 - \lambda) \text{ or } \lambda = \frac{q_{n+1}^* - 1}{q_{n+1}^*}.\]

(9)

Then, the constraint (8) becomes

\[(q_1/q_{n+1}^*, \ldots, q_n/q_{n+1}^*, \lambda) \leq (q_1/q_{n+1}^*, \ldots, q_n/q_{n+1}^*, 1)A.\]

(10)

The constraint in this form corresponds to eq.(3), giving the value of each commodity as

\[\lambda_j = q_j^*/q_{n+1}^* \text{ for } j = 1, \ldots, n,\]

(11)

where \(q_j^*\) is the \(j\)-th element of the optimum vector \(q^*\) found by solving the problem (DL). Told differently, the constraint in (DL) indirectly requires that the labour value of each commodity be less than or equal to the amount of labour which has entered that commodity directly or indirectly. We may call \(q^*\) “optimal gross shadow values”.

In sum, in order to compute the labour values of commodities for a simple Leontief model, we first solve the linear programming problem (DL), and derive the final results using eqs.(9) and (11). Indeed in the simple Leontief system, we can solve the problem in a straightforward way as

\[q'' = e'_{[n+1]}(I - A)^{-1}.\]
It is unnecessary to use the linear programming problems in the Leontief model above. And yet, we shall find our indirect method useful in more general models. It is noted that the problem (PL) is auxiliary, and has nothing to do with a programming problem in the real world. In fact, based upon our definition of values in section 1 we can give a direct interpretation of the problem (DL), with no recourse to (PL), after a manipulation of inequalities (here equations) as is seen in eq.(9). More detailed explanation shall be presented in section 4 for a general model.

3. VALUES IN A SPECIAL MODEL OF JOINT PRODUCTION
In this section, we deal with a special case in which the output matrix is not the identity matrix, but an \( n \times n \) square nonnegative matrix \( B \), and define

\[
\mathbb{B} \equiv \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.
\]

We continue to assume the absence of bads. Our basic assumption here is:

**Assumption (AS) (Inverse Positivity assumption):**

\[
(B - A)^{-1} = (B - A)^{-1} + (B - A)^{-1} \cdot c \cdot S^{-1} \cdot \ell \cdot (B - A)^{-1},
\]

where

\[
S \equiv 1 - \ell \cdot (B - A)^{-1} \cdot c,
\]

i.e., the Schur complement for the special case here.

For the above assumption to hold, the condition that \( S > 0 \), i.e., the labour value of one unit of labour is less than unity, is necessary, while the conditions that \((B - A)^{-1} > 0\) and \( S > 0 \) are sufficient. Now it is not difficult to see that the labour value of commodities can be obtained first by solving

\[
q'' = c'_{n+1} (\mathbb{B} - \mathbb{A})^{-1},
\]

and by utilizing eqs.(9) and (11) in the previous section. This is because the labour value of a commodity is the amount of labour which has got into that commodity directly or indirectly, and we simply apply the principle of algebra to reach the basic equation like eq.(3) or eq.(4). More precisely, eq.(4) here becomes

\[
\lambda \begin{array}{cc} \lambda & 1 \\ \lambda & 1 \end{array} \mathbb{B} = \begin{array}{cc} \lambda & 1 \\ \lambda & 1 \end{array} \mathbb{A} + (0, \ldots, 0, 1 - \lambda),
\]

and the two linear programming problems are:

**PS**  \( \min x_{n+1} \) subject to \( \mathbb{B}x \geq \mathbb{A}x + c'_{[n+1]} \) and \( x \in \mathbb{R}^{n+1}_+ \),

**DS**  \( \max q_{n+1} \) subject to \( q' \mathbb{B} \leq q' \mathbb{A} + c'_{[n+1]} \) and \( q \in \mathbb{R}^{n+1}_+ \).
Thus, so long as we have a square system and the assumption AS is satisfied, not much is different from a simple Leontief model, confirming a proposition stated in Schefold (1978). (See also Ekuni and Fujimoto (2003).)

Now what Sraffa had in mind when he wrote the sections 66 and 67 in Sraffa (1960, pp.56-58) is a still more special case of the one discussed here. That is, when the matrix \((B - A)\) is transformed by an \(n \times n\) nonnegative matrix \(P\) to an \(M\)-matrix (a inverse-positive matrix with its off-diagonal elements being nonpositive), then \((B - A)^{-1} > 0\), because

\[(B - A)P = M \quad \text{with} \quad P \geq 0 \implies (B - A)^{-1} = PM^{-1} \geq 0.\]

Hence, we have \((B - A)^{-1} > 0\), satisfying our assumption AS.

It is easier to understand the matter if we raise a numerical example of three commodities here. Let

\[B - A = \begin{pmatrix} 1 & 1 & -1.2 \\ -1.2 & 1 & 1 \\ 1 & -1.2 & 1 \end{pmatrix}.\]

We instantly notice that by combining the first two processes with an equal weight we have a positive net output only for the first commodity with the other ones being negative. Similar combinations between other two processes yield again a positive net product only for one commodity. More concretely, this fact is expressed as

\[(B - A) \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1.2 \\ -1.2 & 1 & 1 \\ 1 & -1.2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -0.2 & -0.2 \\ -0.2 & 2 & -0.2 \\ -0.2 & -0.2 & 2 \end{pmatrix}\]

In fact,

\[(B - A)^{-1} = \frac{1}{44} \begin{pmatrix} 25 & 5 & 25 \\ 25 & 25 & 5 \\ 5 & 25 & 25 \end{pmatrix} \geq 0.\]

Punzo (1999) noted that Sraffa adopted the classical method to solve, at least partially, the problem of joint production by “adding processes”. This seems to be, however, different from our interpretation of what is explained in Sraffa (1960). Punzo’s suggestion that the nonsubstitution theorem can remain valid in a special model of joint production discussed here was already confirmed by Dasgupta and Sinha (1979, 1992).\footnote{See also Fujimoto et al.(2003) for a nonsubstitution theorem in a slightly more general model.}
4. VALUES IN A GENERAL MODEL

The original model in von Neumann (1945-46) assumes that workers’ consumption basket is fixed and one unit of homogeneous labour is paid before production one and the same basket. In this von Neumann model, labour is fused into material inputs inseparably so that no labour value theory is possible. Morishima (1964) takes out the labour input coefficient vector, and in Morishima (1973, p.185) he proposed the concept of “optimal values”, $\lambda$, using a linear programming problem:

\[
\text{(DM)} \quad \max \lambda c \quad \text{subject to} \quad \lambda B \leq \lambda A + \ell \quad \text{and} \quad \lambda \in \mathbb{R}^n_+,
\]

where $B$ and $A$ are now the $m \times n$ rectangular output coefficient matrix and the input coefficient matrix, respectively, $\ell$ the $n$-row vector of labour input coefficients for $n$ production processes. (See also Morishima (1974).) We should notice that even in Leontief models “inefficient” processes are hidden behind the scene, and a programming problem is latent. In this formulation, symmetry is lacking. That is, while there can be many processes to produce a commodity, one single household activity is given to reproduce labour power. Besides, while durable capital goods are taken into consideration among production processes, no durable consumption goods are admitted in the household.

Our model allows for these elements as well as many types of heterogeneous labour, teachers, computer programmers, unskilled manual workers, and so on. (See Krause (1981) and Fujimori (1982) for some treatments of heterogeneous labour.) Indeed, many types of labour come into our model just like ordinary commodities, enjoying a plural number of alternative household activities to reproduce a particular type of labour with durable consumption goods involved. Some amount of labour may be required to produce a type of labour in a household. We do not need any efficiency conversion rates among various types of labour. Since various types of labour are exactly like normal commodities, we can use the same symbol $B$ and $A$, as the output and input coefficient matrices, both of which now have $m$ rows and $n$ columns. There are altogether $m$ kinds of goods, services, various types of labour, and now “bads” in this section. Old machines are included in line with fictitious intermediate goods to realize the uniform period of production or activities. Likewise, a durable consumption commodity in a column of $A$ will appear in the corresponding column of $B$ as one period, say one year, older commodity.

Some words are in order about land. Various types of lands or Ricardian lands are not excluded in our general model, and can be treated just like durable machines. (See Kurz (1986) for von Thünen’s treatment of land.) Arable land as well as residential areas can be reclaimed from the sea or the forests. It is important to note that when we consider values in whatever unit, ‘what belongs to whom’ does not matter. When we discuss about prices, the social system of property right comes onto the stage with such categories like prices themselves, rents and interests.
Then, we can choose any commodity or a type of labour as the standard of value because goods and labour are treated in a completely symmetric way so far as values are concerned. (We have not yet introduced prices, on which profits have an influence in an asymmetric manner.) One can compute the petroleum (or crude oil) values of commodities and those of various types of labour. The \(i\)-th commodity (or labour) values of various commodities are obtained through first solving the following linear programming problem:

\[
(DG) \quad \max q_i \quad \text{subject to} \quad q' \mathbb{B} \leq q' \mathbb{A} + b^{(i)} \quad \text{and} \quad q' \in \mathbb{R}_+^m,
\]

where \(b^{(i)}\) is the \(i\)-th row of \(\mathbb{B}\). Next, using the optimal solution row \(m\)-vector \(q^*\), we can calculate the values as

\[
\lambda_i^{[i]} = \frac{q_i^* - 1}{q_i^*} \quad \text{and} \quad \lambda_j^{[i]} = q_j^*/q_i^* \quad \text{for} \quad j = 1, \ldots, m, \ j \neq i,
\]

where \(\lambda_j^{[i]}\) stands for the \(i\)-th commodity value of \(j\)-th commodity. These equations correspond to eqs.(9) and (11). It is easy to verify that if there is only one type of homogeneous labour, which is the \(m\)-th commodity, and this labour is reproduced by the \(n\)-th activity, and there is no durable consumption goods, then \(\mathbb{B}\) and \(\mathbb{A}\) are of the following special form

\[
\mathbb{B} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{A} = \begin{pmatrix} A & c \\ \ell & 0 \end{pmatrix},
\]

which makes the problems (DG) designating labour as the numeraire coincide with Morishima’s problem (DM).

The programming problem dual to the above (DG) is:

\[
(PG) \quad \min b^{(i)} x \quad \text{subject to} \quad \mathbb{B} x \geq \mathbb{A} x + e^{[i]} \quad \text{and} \quad x \in \mathbb{R}_+^n,
\]

where \(e^{[i]}\) is the \(m\)-column vector whose \(i\)-th entry is unity with all the remaining elements being zero.

Therefore, all we have to assume in our general model is

**Assumption (AG) (Productiveness assumption):** There exists an \(x \in \mathbb{R}_+^n\) such that

\[
(\mathbb{B} - \mathbb{A}) x \succ 0.
\]

Or if we stick to a particular commodity \(i\) as the standard of value, the above assumption can be made weaker as

**Assumption (AG2) (Productiveness assumption 2):** There exists an \(x \in \mathbb{R}_+^n\) such that

\[
(\mathbb{B} - \mathbb{A}) x \geq e^{[i]}.
\]

Given one of these assumptions, the set of feasible vectors for the programming problem (PG) is not empty, and \(b^{(i)} > 0\), hence the problem (PG) has an optimal solution. It should be remembered that whatever commodity is chosen
as the standard, its own value is uniquely determined, while the values of the remaining ones may not be unique. It may be added that the diameter of the set of optimal vectors to (DG) may shrink as the number of processes gets larger.

At this point, it is also important to note that by the very formulation of the constraint in the problem (PG), the optimum value $b^{(i)} \cdot x^* \geq 1$, thus by duality $q^*_i \geq 1$, leading to $0 \leq \lambda_i^{[i]} < 1$. Thus, we have

**Proposition 1.** Given the assumption (AG) or (AG2), the $i$-th commodity value of $i$-th commodity is less than unity.

This proposition is misunderstood by some people as expressing the exploitation of commodities. (See Bowles and Gintis (1981) and Roemer (1982, 1986). Actually this has nothing to do with exploitation, but an alternative expression of productiveness of an economy as was shown in Fujimoto and Fujita (2006).

Now we can give a precise definition of values in a retrospective way.

**Definition.** Values are nonnegative magnitudes assigned to commodities (including services and various types of labour) such that the value of the standard commodity be maximized under the condition that the total value of the output of each possible process should not exceed that of the input. When calculating the total value of the input of a process, unity is assigned to the direct input of the standard commodity.

Having defined values in this way, we can now explain how the problem (DG) comes out of this definition, which shows why our definition of the $i$-th commodity values of commodities is a natural and proper generalization of the concept shared by the classical economists. To do so, let us first define the following vectors:

$$\Lambda^{[i]} \equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \ldots, \lambda_{i-1}^{[i]}, \lambda_i^{[i]} \cdot 1, \lambda_{i+1}^{[i]}, \ldots, \lambda_m^{[i]}),$$

$$\Lambda_i^{[i]} \equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \ldots, \lambda_{i-1}^{[i]}, 1, \lambda_{i+1}^{[i]}, \ldots, \lambda_m^{[i]}).$$

The vector $\Lambda^{[i]}$ is the vector of values with $i$-th commodity being the standard of value. Our definition above is rewritten like this:

**Find out $\Lambda^{[i]} \geq 0$ such that $\lambda_i^{[i]}$ should be maximized**

subject to $\Lambda^{[i]} \cdot B \leq \Lambda_i^{[i]} \cdot A$.

The constraint in this problem is further transformed first through multiplying both sides by nonzero nonnegative $v$, then adding $(1 - \lambda_i^{[i]}) \cdot b^{(i)}$ to both sides, which results in

$$v \cdot \Lambda^{[i]} \cdot B \leq v \cdot \Lambda_i^{[i]} \cdot A + v \cdot (1 - \lambda_i^{[i]}) \cdot b^{(i)}.$$ 

Then, we set

$$v \cdot (1 - \lambda_i^{[i]}) = 1 \text{ or } \lambda_i^{[i]} = 1 - \frac{1}{v}.$$
This normalization yields as our constraint
\[ v \cdot \Lambda_{[i]} \cdot B \leq v \cdot \Lambda_{[i]} \cdot A + b^{(i)}. \] (13)
Since we have \( \lambda^{[i]} = 1 - 1/v \) from our normalization, maximizing \( v \) is equivalent to maximizing \( \lambda^{[i]} \). Writing \( v \cdot \Lambda_{[i]} \) simply as a variable vector \( q \), we have the linear programming problem (DG). This derivation itself clearly shows that the problem (PG) on quantity side is after all auxiliary.

Whatever commodity or service can serve as a standard of value. Smith (1776, Ch.5) argued that labour of an abstract nature may have an advantage over others to be a standard because it does not vary much in its own value. This gives only a partial answer. Smith added the historical fact as one more reason that labour amount required to obtain a commodity was counted upon to decide exchange rates among goods in “that early and rude state of society” before a sort of money was implemented and before the accumulation of capital stock was significant. We wish to include another reason put forward by Hilferding (1904) that labour as the whole entity is a glue or a net which combines various atomistic factors into a social activity.4

We give one more reason why we should choose labour as the standard of values in section 7 on prices. And yet, certainly on some occasions, people wish to know the petroleum value of commodities. In that case, our definition can be a guidance even in various restrictive models.

To conclude this section, we explain why we consider the magnitudes \( \lambda_i \)'s obtained through the problem (DG) and eqs.(12) as the values in terms of commodity \( i \), and not the “optimum values” following Morishima (1973, 1974). Morishima distinguishes optimum values from “actual values” which are calculated based on those processes actually in use. Actually used processes are, however, changed through the shift in profit rates or more generally through price movements. When we wants to establish the concept of value before distribution is discussed, we should take into consideration the entire set of possible processes, and this necessitates the explicit or implicit formulation of a programming problem which involves the whole set of processes as a production system. Thus we call our magnitudes \( \lambda_i \)'s here simply “values”. It is noted that even in a simple Leontief model without any choice of processes, a hidden linear programming can be conceived as is shown in section 2. In Leontief models, all inefficient processes are, in fact by assumption, clearly distinguished and left out at the outset.

5. BADS AND DISPOSAL PROCESSES

The reader may be wondering where bads and their costly disposal processes are seen in our general model of section 4. They are there already, and treated

4In his words: “Weil also die Arbeit das gesellschaftliche Band ist, das die in ihre Atome zerlegte Gesellschaft verbindet, und nicht weil sie die technisch relevanteste Tatsache ist, ist sie Prinzip des Wertes und besitzt das Wertgesetz Realität.” (Hilferding (1904).)
in the same way as goods and services so that they are not conspicuous. One numerical example will help. Let us suppose there is one ordinary commodity called corn, a pollutant called manure, and only one type of labour, and that there exist three processes or activities. The technical data are:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 1 \\
0 & 10 & 0 \\
1 & 1 & 0 \\
\end{pmatrix}
\]

The three rows represent the above three in that order. Then, after designating labour as our standard, i.e., \(i = 3\). The problems (PG) and (DG) become

\[
\begin{align*}
\text{(PG)} & \quad \min b^{(3)} x = x_3 \quad \text{subject to} \quad B x \geq A x + (0, 0, 1)' \quad \text{and} \quad x \in \mathbb{R}_+^n, \\
\text{(DG)} & \quad \max q_3 \quad \text{subject to} \quad q' B \leq q' A + (0, 0, 1) \quad \text{and} \quad q' \in \mathbb{R}_+^m.
\end{align*}
\]

Then, we get a unique solution vector \(x^*\) and a \(q^*\) respectively described as

\[
x^* = (1/9, 0, 10/9) \quad \text{and} \quad q^* = (1/9, 0, 10/9).
\]

The formulae (12) yield

\[
\lambda^{[3]} = (1/10, 0, 1/10).
\]

The labour value of manure is zero, and the disposal process is not activated since there is no social or governmental intervention.

Next, we choose one unit of manure as the standard of value, i.e., \(i = 2\). In a similar way to the above, we get

\[
x^* = (1/10, 0, 1) \quad \text{and} \quad q''^* = (0, 1, 0), \quad \text{and so} \quad \lambda^{[2]} = (0, 0, 0).
\]

These results seem natural though the reader may insist a negative value for manure, and bring the input of manure in the second process to the output side. Whether to operate a process or not, however, is determined by consideration of profit, a phenomenon in the world of prices not values. Certainly after discussing prices, we may introduce “revised values”, allowing for negative values in some commodities which carry negative prices.

To show that our model can accommodate heterogeneous labour, we raise an example here in passing, by including skilled labour in the preceding one. The third rows of the following matrices are the same as above and here called unskilled or simple labour, while the fourth rows stand for skilled labour requiring unskilled labour in its reproduction as domestic service and alike.

\[
\begin{pmatrix}
11 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 10 & 0 & 0 \\
1 & 1 & 0 & 0.1 \\
0.2 & 0.1 & 0 & 0 \\
\end{pmatrix}
\]

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When we choose simple labour as the standard, the problem (DG) becomes

\[
(DG) \quad \max q^3 \quad \text{subject to } q^3 B \leq q^3 A + (0, 0, 1, 0) \quad \text{and } q^3 \in R^m_+,
\]
from which we obtain

\[
q^{**} = (0.119, 0, 1.119, 0.350), \quad \text{and so } \lambda^{[3]} = (0.106, 0, 0.106, 0.313).
\]

Next, when skilled labour becomes the standard, the problem (DG) is

\[
(DG) \quad \max q^4 \quad \text{subject to } q^4 B \leq q^4 A + (0, 0, 0, 1) \quad \text{and } q^4 \in R^m_+,
\]
whence follow

\[
q^{**} = (0.023, 0, 0.023, 1.049), \quad \text{and so } \lambda^{[4]} = (0.022, 0, 0.022, 0.047).
\]

It should be noted that the ratios between two values of two objects are in general different from each other when the standards are different.

Surely our model enables us to deal with far more general cases in which various types of skilled labour, through education and training processes, are produced from unskilled and/or skilled labour. How to define less skilled labour is given in some more details in the following section.

6. NECESSITIES AND LUXURIES

In this section, we briefly discuss in a quantitative way what are luxury goods as opposed to necessities. Let us include a commodity in the example model used at the end of the previous section, which commodity is consumed only by skilled labour or managerial directors. The technical data are:

\[
B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0.9 \\
0 & 11 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 10 & 0 & 0 \\
10 & 1 & 1 & 0 & 0.1 \\
1 & 0.2 & 0.1 & 0 & 0
\end{pmatrix}
\]

The first row and the first column are added to the last numerical example. The first row represents a would-be luxury commodity, while the first column stands for a normal production process of that commodity. The remaining rows represent the same stuff in the same order, i.e., the second row for corn, the third for manure, the fourth for unskilled labour, and the last for skilled labour. To produce one unit of the first commodity, 10 units of unskilled labour and

---

\footnote{When we deal with prices, the fact that skilled labour is created from unskilled one through production makes the model too much complicated. In this paper, we do not deal with heterogeneous labour as stocks, and simply assume each type of labour is abundantly available. In principle, heterogeneous labour can be treated as various vintages of capital goods.}
one unit of skilled labour are required. To reproduce one unit of skilled labour, one unit of the first commodity is consumed. The first commodity is durable and only one tenth depreciate away in a period. When we choose the unskilled labour as the standard, we obtain

\[ q^{**} \div (13.235, 0.150, 0, 1.150, 1.738), \text{ and so } \lambda^{[4]} \div (11.509, 0.130, 0, 0.130, 1.511). \]

Next, when we choose the skilled labour as the standard of value, we have

\[ q^{**} \div (1.471, 0.0267, 0, 0.0267, 1.203), \text{ and so } \lambda^{[5]} \div (1.223, 0.0222, 0, 0.0222, 0.169). \]

Looking at these figures, especially noting the relative sizes between \( \lambda^{[4]} \) and \( \lambda^{[5]} \) (and between \( \lambda^{[5]} \) and \( \lambda^{[5]} \)), we can reaffirm the fourth commodity as unskilled labour: the smaller, the more “primary” provided that they are measured in the common unit, e.g., person-hour. In sum, this suggests that a type of labour may be defined as “less skilled” when the labour value is smaller in terms of a certain type of labour. Almost needless to say, ambiguous cases take place just like double switching.

We proceed to the problem of how to define necessities and luxuries for unskilled labour. To do so, let us regard the reciprocal of labour values of labour, here \( \lambda^{[4]} \), as an index of real wage rate, about which more detail is given in the next section on prices. Intuitively, when the elasticity of the real wage rate with respect to the requirement of unskilled labour to produce one unit of a commodity is large enough, that commodity is a necessity, while the elasticity is so small, the commodity is a luxury. Suppose the input coefficient of unskilled labour in the corn production process 2 decreases by 50% from the present unity to 0.5, the solution with the unskilled labour as the standard becomes

\[ q^{**} \div (12.358, 0.0845, 0, 1.0845, 1.513), \text{ and so } \lambda^{[4]} \div (11.395, 0.0779, 0, 0.0779, 1.395). \]

Thus, the real wage rate index increases from 1/0.130 to 1/0.0779, i.e., 66% increase, obtaining a rough estimate of elasticity 66/50 = 2.32. Next suppose that the input coefficient of unskilled labour in the luxury production process 1 decreases by 50% from the present 10 to 5, the solution with the unskilled labour as the standard becomes

\[ q^{**} \div (6.728, 0.135, 0, 1.135, 1.055), \text{ and so } \lambda^{[5]} \div (5.928, 0.119, 0, 0.119, 0.929). \]

This time, the real wage rate index increases from 1/0.130 to 1/0.119, i.e., only 9% increase, obtaining an estimate of elasticity 9/50 = 0.18. Hence our intuition has worked, and this elasticity may be used to classify the commodities as a luxury or not, without using demand theory or income data.
7. PRICES

“Values” are concerned with the strictly technical aspects of production and have nothing to do with market inducements. Sraffa (1960) initially tried to build precisely an inducement-free theory of relative prices. As soon as he realised, however, that the existence of a surplus makes relative prices dependent upon income distribution, he replaced “reproductive consumption” of workers with a variable real wage, in accordance with the Ricardian theory, that wage was inversely related to a uniform rate of profit. In order to determine relative prices in this classical framework, one has to start from either the rates of profit or the real wage rates as given. We pick up the rates of profit, assume the i-th commodity is i-th type labour which is unskilled and cannot be produced in normal production processes, and we assume wages are paid before production, while Sraffa assumes post factum payment. Then consider the following linear programming problem:

\[(DGP) \quad \max q_i \quad \text{subject to} \quad q' B \leq q' A \cdot R + b^{(i)} \quad \text{and} \quad q' \in R^m_+ ,\]

where \( R \) is a given \( n \times n \) diagonal matrix whose \((j, j)\)-element is \( 1 + r_j \), \( j = 1, \ldots, n \). When \( j \)-th process is a household activity to reproduce a type of labour, the rate of profit \( r_j = 0 \). It is convenient to define a subset \( H \) of the index set \( N \equiv \{1, 2, \ldots, n\} \) so that \( H \) consists of the indexes which represent household activities. Thus, when \( j \in H \), \( r_j = 0 \). If we consider an equilibrium in a competitive economy, \( r_j = r \) for all \( j \notin H \). Let a solution vector of the problem (DGP) be \( q^* \), and calculate the prices with the numeraire being labour \( i \) as

\[ p^{[i]}_i = \frac{q^*_i - 1}{q^*_i} \quad \text{and} \quad p_j^{[i]} = q^*_j / q^*_i \quad \text{for} \quad j = 1, \ldots, m, \ j \neq i, \quad (14) \]

which corresponds to formula (12) in section 4. As in section 4, we can explain why the formula (14) represents the prices in a general model. In the constraint of (DG) at an optimal \( q^* \), transfer the term \( b^{(i)} \) to the left-hand side, and divide the whole inequalities by \( q^*_i \), then the constraint becomes

\[ p^{[i]} B \leq p^{[i]} A \cdot R , \quad (15) \]

where \( p^{[i]}_i \) is the price vector with its \( i \)-th entry replaced by unity as \( \Lambda^{[i]}_i \) in section 4. This simply means that one unit of \( i \)-th labour is the numeraire. This transformed constraint tells us that each ordinary (goods and services) production process \( j \) cannot make super normal profits greater than a given rate \( r_j \). Thus, we may define long-run competitive wage-unit prices as follows:

**Definition.** Long-run competitive prices are nonnegative magnitudes assigned to commodities (including services and various types of labour) such that the price of the standard commodity be maximized under the condition that the total value of the output of each possible process should not exceed the total cost of the input augmented with a uniform profit rate. When calculating the
total cost of the input of a process, unity is assigned to the direct input of the standard commodity.

What is not a problem in the definition of values could be a problem here. That is, the reader may object our interpretation of (15) because on the left-hand side $p_t^{[i]}$ is used while on the right-hand side unity is used for the numeraire to calculate the cost. This is like the labour value of labour, and not easy to swallow instantly. The $i$-th labour is, however, unskilled one not produced in ordinary production processes by our definition, thus $p_t^{[i]}$ never enters the calculation of the sales of each ordinary process. It appears only in the household activities which produce $i$-th labour. Therefore we can regard the reciprocal of $p_t^{[i]}$ as an index of the real wage rate for the $i$-th labour. When we choose a commodity which is produced by ordinary production processes, the above interpretation fails. This may give another reason why we had better select labour as the standard of value at the outset.

The programming problem dual to (DGP) is:

$$(\text{PGP}) \min b^{(i)} x \text{ subject to } \mathbb{B} x \geq A \cdot R \cdot x + e[i] \text{ and } x \in \mathbb{R}^n_+.$$ 

When we consider a competitive equilibrium, i.e., when $r_j = 7$ for all $j \not\in H$, 7 may be interpreted as a uniform growth rate among ordinary processes involving all the goods and services, and labour employed in production processes.

Looking back, we should have made in this section

**Assumption (AGP) (Productiveness assumption):** There exists an $x \in \mathbb{R}^n_+$ such that 
$$(\mathbb{B} - A \cdot R)x \gg 0.$$ 

If we stick to labour $i$ as the numeraire, the following still weaker assumption is enough.

**Assumption (AGP2) (Productiveness assumption 2):** There exists an $x \in \mathbb{R}^n_+$ such that 
$$(\mathbb{B} - A \cdot R)x \geq e[i].$$ 

Now let us discuss about bads. Just as in the case of value calculations, no disposal processes are operated unless some intervention is made. When the government designate a subset of commodities as bads, and they should have negative prices, the corresponding rows of $B$ and $A$ are interchanged. With the uniform rate of profit being 20%, the numerical example in section 5 becomes

$$\mathbb{B} \equiv \begin{pmatrix} 11 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A \equiv \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \text{ and } R = \begin{pmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{Unskilled labour may be produced as joint products in a household activity.}$$

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Solving the problem (DGP) with \( i = 3 \), i.e., labour, gives \( q_1^* = 150/928 \), \( q_2^* = 147/928 \), \( q_3^* = 1225/928 \). Therefore, the prices are: \( p_1^* = 150/1225 \), \( p_2^* = 147/1225 \) and \( p_3^* = 297/1225 \). The index of real wage rate is \( \omega \equiv 1/p_3^* = 1225/297 = 4 \). One unit of labour is paid one unit of money, and using this the owner of this labour could reproduce about 4 units of labour if all the wages were consumed and the household activity were operated at the intensity level 4. A household activity is accompanied with a payment for disposal of manure (commodity 2) before its operation, while it is assumed that the owner of disposal process gets paid at the end of production period. When these timings of payments differ from our postulate, we have to adjust the profit rate, or the mark-up ratio accordingly. Here, \( p_2^* \) is regarded as showing an absolute magnitude of a negative price.

The example with heterogeneous labour in section 5 becomes as follows:

\[
\begin{bmatrix}
11 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
1 & 1 & 0 & 0.1 \\
0.2 & 0.1 & 0 & 0
\end{bmatrix}
\]

Setting the profit rate 0, and selecting unskilled labour as the standard, we obtain

\[ q'' = (0.141, 0.134, 1.275, 0.678) \], and so \( \lambda = (0.111, 0.105, 0.216, 0.532) \).

Since the second abatement process is utilized, labour values of two types of labour get larger than when manure is left unprocessed.

We can establish the following proposition.

**Proposition 2.** When the assumption (AGP) or (AGP2) is valid, a higher uniform rate of profits corresponds to a lower real wage rate of \( i \)-labour provided that \( i \)-labour value of \( i \)-th labour is positive.

**Proof.** Consider an optimal solution \( x^* \) to (PGP) at a certain uniform profit rate \( r \). When \( i \)-labour value of \( i \)-th labour is positive, the minimum value \( b^{(i)}x^* \) is greater than unity. This implies the \( i \)-th element of \( (A \cdot R \cdot x^*) \), i.e., \( (A \cdot R \cdot x^*)_i \), is positive. Because of constant returns to scale, this \( i \)-th constraint is binding. Thus, increasing a given profit rate \( r \) makes any optimal solution infeasible. This implies a greater optimal value in the new situation with a higher profit rate. By the duality theorem in linear programming, the maximal value \( q_i \) in (DGP) should also an increase. Then, in view of the definition of \( p_{[i]} = (q_i^* - 1)/q_i^* \) the real wage rate is the reciprocal of \( p_{[i]} \), it has to fall.\( \square \)

This proposition is contradictory to what is stated in Kurz (1978, p.34), that is, a higher profit rate together with a higher real wage rate can be realized in an equilibrium with a lower ground rent. This is because in Kurz (1978) the simultaneous use of two processes of different intensity is possible irrespective of their efficiency. The only requirement is that those two processes yield the same rent and the same profit rate. Clinging to the equality system can lead
to inefficiency. On the other hand, our model has nothing to do with demand, and once the rate of profit is given, the set of efficient processes is determined, a long-run phenomenon. Now we have to explain how to grasp rent in our model.

Now the reader may wonder why in the real world mutton and wool are both keeping positive prices while the mechanical determination of equilibrium prices by eq.(14) is likely to assign zero to one of their prices. The answer is simple. This is because there exist at least one normal production process which can make supernormal profits, if one of their price is zero and all the other prices are set to the optimal solutions.

8. RENT

Our model can include land provided it is produced like machines by reclaiming from the sea or the forests, and can also deal with an isolated state a la von Thünen with explicit transportation costs. One thing we note is that when the private property right is set up, we should distinguish two methods of production, one by buying machines or land, and the other by borrowing them or buying the services rendered by those properties. Let us consider a numerical example in Fujimoto et al. (2003) where there are two perishable commodities and one durable machine which can last physically for two periods, and is discarded with no scrap value. The technical data are given as follows. The first two rows stand for two perishable goods, the third row is the brand new machine, the fourth row corresponds to the one year old machine, and the last row represents labour. There are five normal production processes, and one household activity to reproduce labour by consuming the first commodity. The first two processes or columns are carried out by buying the machine, i.e., the third commodity, and keeping the one year old machine in the following period.

\[
B \equiv \begin{pmatrix}
1 & 1 & 1 & 0 & 0.2 & 0 \\
1 & 1 & 1 & 0 & 0.2 & 0 \\
0 & 0 & 1 & 1 & 0.2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\text{ and } A \equiv 
\begin{pmatrix}
0 & 0.2 & 2 & 0 & 0 & 0.1 \\
0 & 0.1 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

For simplicity, we assume the rate of profit is zero, and adopt labour as the standard of value or the numeraire. Then we have the problem

\[(DGP) \quad \max q_5 \text{ subject to } q^' B \leq q^' A + (0, 0, 0, 0, 0, 1) \text{ and } q^' \in R^6_+.
\]

Solving this problem, we have

\[
q^* \equiv (1.250, 1.250, 2.375, 1.000, 1.125), \text{ and so } \\
\lambda \equiv (1.111, 1.111, 2.111, 0.889, 0.111).
\]

At these values, the fifth process makes losses if operated. As the labour input coefficient in the second process which uses the old machine, i.e., \( \lambda_{52} \), becomes
larger, the second process gets less profitable. When \( A_{52} \equiv 2.54 \), the second process ceases to be used, while the fifth has become viable.

Now we consider the case in which tenant-capitalists borrow the machines. So, we add two more processes, the sixth and seventh columns. The sixth process is operated by buying the machine and lend that new machine to another producer, creating the service of the new machine, now represented in the top row. The seventh process is in a similar way, to create the service of the old machine by keeping that machine. The technical data and the resulting solutions are as follows.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\
1 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0.1 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
q'' \div (1.375, 1.000, 1.250, 1.250, 2.375, 1.000, 1.125), \quad \text{and so} \quad
\lambda \div (1.222, 0.889, 1.111, 1.111, 2.111, 0.889, 0.111).
\]

Naturally, in this simple case, the sum of the rental of the brand new machine and that of the old one is equal to the price (or value) of the new machine: \( 1.222 + 0.889 = 2.111 \). The point is that our model can deal with more realistic complicated cases in a systematic way.

Rents can be dealt with as rentals for durable machines can. In the short run, however, we may have to consider the existence of differential rents. These rents can be computed as follows. First, we compute the prices and the rents by using the problem (DGP) in the preceding section. Next, remove the efficient processes which produce a particular commodity and at the same time involve the land. Then, compute again the prices and rents. Finally, restore the removed processes with differential rents added on the cost side. By repeating this procedure, we can find multiple ranks of fertility with their associated rents.

Let us give a numerical example. In this example, there are two normal commodities, corn and wine, and one type of homogeneous labour. There are also three kinds of lands of different fertility. These lands can last for ever, and continue to render service by the help of labour. The technical data are:

\[
\begin{pmatrix}
4 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0.2 & 0.2 & 0.2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
The first two rows of these matrices show corn and wine, and the third stands for labour with the remaining three rows for services by three types of land. The first three columns represent the production processes of corn using three types of land, the fourth for wine, the fifth for labour, and the last three columns stand for the processes to yield land service and maintain land. Land itself does not appear explicitly because land is assumed to last for ever with an amount of maintenance work. Now the solutions turn out

\[ x^* = (1.25, 0.0, 0.0, 2.5, 5.0, 1.25, 0.0, 0.0), \]
\[ q^{**} = (1.5, 2.5, 5.0, 1.0, 0.0, 0.0), \text{ and so} \]
\[ \lambda = (0.3, 0.5, 0.8, 0.2, 0.0, 0.0). \]

Thus, the labour value of land service of the first type is 0.2, others being null. Now leave out the first process, and we get

\[ x^* \div (*, 3.333, 0.0, 5.0, 10.0, 0.0, 3.333, 0.0), \]
\[ q^{**} = (4.0, 5.0, 10.0, 0.0, 2.0, 0.0), \text{ and so} \]
\[ \lambda = (0.4, 0.5, 0.9, 0.0, 0.2, 0.0). \]

When we operate the first process, we have an extra \( 0.6 = 4 \times 0.4 - 1 \times 1 \), making the value of land service of the first type equal to 0.6 which is greater than that for the second type 0.2. In such a simple single production case it is evident that land of “higher” quality yields greater rent for its service because of the very nature of linear programming.

9. INTERNATIONAL TRADE: VALUES AND PRICES

As in the real world, we have to introduce international trade to our model. An important point to remember is that while goods and services are transported among countries, people in general are not allowed to migrate by rule or by assumption. There are some methods conceivable, and the one of the easiest is to include, as production processes, possible vectors of imports (positive entries) and exports (negative entries), or just the statistical average of the past few years’ imports and exports. Let us consider a small country with two commodities, corn and wine, both produced by homogeneous labour, here called unskilled labour or simply labour. The technical data without international trade are as follows:

\[ B - A \equiv \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \]

\(^7\)The authors are grateful to Professor Sergio Parrinello for his pointing out our confusion between values and prices, and our errors in this section about theory of comparative advantage.
The first row represents corn, the second does wine, and the last simple labour. The last column describes the only household activity to produce labour. The solutions of (PG) and (DG) are
\[ x^* = (0.6, 0.8, 2.4), \]
\[ q^* = (0.6, 0.8, 2.4), \] and so
\[ \lambda \div (0.25, 0.333, 0.58333). \]

When we add an process of import and export, \((-1, 1, 0)\), that is, exporting one unit of corn in exchange for one unit of wine imported, the technical data become
\[ \mathbb{B} - \Lambda \equiv \begin{pmatrix} 4 & 0 & -1 & -1 \\ 0 & 3 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}. \]
The solutions are now
\[ x^* = (1.0, 0.0, 2.0, 2.0), \]
\[ q^* = (0.5, 0.5, 2.0), \] and so
\[ \lambda \div (0.25, 0.25, 0.5). \]

These results show that by international trade, one unit of corn for one unit of wine, this economy specializes in the production of corn, and through trade, the labour values decrease or remain unchanged. One can add another process, \((1, -1, 0)\), that is, exporting one unit of wine in exchange for one unit of corn, but this causes no change to the above results because the process is not used by the economy. In this model of a small open economy, the terms of trade, expressed here in vector form, \((-1, 3/4, 0)\) or \((1, -3/4, 0)\), is the switch point, across which what is exported is interchanged for what is imported.

Likewise, a two country model can be constructed. For example, consider two countries England and Portugal, two commodities, again corn and wine, and homogeneous unskilled labour in each country. The technical data are:
\[ \mathbb{B} - \Lambda \equiv \begin{pmatrix} 4 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 3 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 3 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \]
The first two rows are corn and wine for England, the next two rows are for Portugal, the fifth row shows labour force of England, and the last row is for that of Portugal. The last two columns represent possible international trade vectors with the price ratio being unity. We have exactly the same solutions as in the above when we solve labour values in terms of English labour unit. In this case
there is, for Portugal, another switch point of terms of trade, again expressed in vector form, \((-1, 3/2, 0)^t\) or \((1, -3/2, 0)^t\). Between the two switch points, the interests of two countries coincide.

The foregoing ways to incorporate international trade need price ratios among commodities. Thus one may insist values depend on prices. Or when we apply these methods to the days of imperialism, we may admit colonial exploitation as a part of “technical data” in order to calculate labour values. We have to avoid such distortions. One intuitive way is to regard unskilled labour of each country as homogeneous, and consider a joint optimization problem in a symmetrical way.\(^8\) In so doing, processes expressing international trade do not need price ratios, but simply show that the export of a commodity is to bring that commodity to the international market, and the import is a reverse operation. That is, export and import are regarded as separate processes. Look at the following matrix.

\[
B - A \equiv \begin{pmatrix}
4 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\end{pmatrix}.
\]

The upper-left \(6 \times 6\) matrix is the same to (16), and the added last two rows mean corn and wine in the international market respectively. The added 8 columns represent the transportation of one unit of commodities from a country to/from one of the international markets.\(^9\) No ratio between two commodities is involved here. We make

**Definition.** Values in our model with international trade are nonnegative magnitudes assigned to commodities (including services and various types of labour) such that the sum of the values of the standard commodity in each country be maximized under the condition that the total value of the output of each possible process should not exceed that of the input. When calculating the total value of the input of a process, unity is assigned to the direct input of the standard commodity in each country.

We adopt the sum of the values as the objective function to be maximized because of symmetry for all the countries. Thus, when the input-output coefficients are completely the same among countries, the same values obtain as in

\(^8\)We assume that one unit of simple labour is produced by the same set of household activities in both countries. Alternatively, we may assume that there exists a common type of labour in each country one unit of which is produced by the same set of household activities.

\(^9\)We here find it easy to allow for non-tradable goods and services, especially services rendered by direct human labour.
the case of isolation. On the other hand, we require that the input of the standard commodity should be assigned unity in each country because otherwise, i.e., zero is assigned in one or more countries, the constraint would be so severe that an optimal value cannot go beyond unity. When looked from the quantity side, if the vector $b$ below contains only one unity, the country corresponding to this sole unity can feed upon the other countries by producing nothing and by importing unilaterally.

Then, the linear programming problem to be solved is a variant of (DG), i.e.,

$$(DG) \quad \text{max} \quad q^t \cdot c \quad \text{subject to} \quad q^t B \leq q^t A + b' \quad \text{and} \quad q^t \in R_+^8,$$

where $b' \equiv (0, 0, 0, 0, 1, 0, 0, 0, 0)$, and $c \equiv (0, 0, 0, 0, 1, 1, 0, 0)$.

Here, $b'$ is actually $b^{(5)} + b^{(6)}$.

The dual to the above is

$$(PG) \quad \text{max} \quad b' \cdot x \quad \text{subject to} \quad Bx \geq Ax + c \quad \text{and} \quad x \in R_+^{14}.$$ 

In sum, in defining labour values, we do not discriminate English labour from Portuguese one, and try to minimize the total labour necessary to produce one unit of labour in both countries. Values can be calculated by the same eqs. (12) for a similar reason in section 4, because we identify English simple labour with Portuguese.$^{10}$ It should be noted again that the problems are set for two countries in a symmetric way.

The solutions of these problems are

$$x^* = (1.2, 1.6, 0, 0, 3.8, 1.0, 1.0, 1.0, 1.0, 0, 0, 0)',$$
$$q^{**} = (0.6, 0.8, 0.6, 0.8, 2.4, 2.4, 0.6, 0.8),$$
$$\lambda \div (0.25, 0.333, 0.25, 0.333, 0.58333, 0.58333, 0.25, 0.333).$$

It seems awkward that values, for England, show no change from the model without international trade, no normal production processes are used in Portugal, Portugal imports corn and wine while exporting nothing, hence there is no gain for England in this virtual minimization problem. On reflection, however, this result is not so strange when the world economy is to be organized with efficiency as is defined above.$^{11}$ When given data allows the nonsubstitution theorem to hold, values will not change even if $c$ assumes whatever nonnegative nonzero vector. Thus, values remain unchanged when we make $c$ as

$$c \equiv (0, 0, 0, 0, 1, 0, 0, 0)'.$$

$^{10}$Consider $\Lambda_{[i,j,\ldots]}$, where unity is set at all the entries for simple labour of each country, and apply the argument leading to eq.(13)).

$^{11}$In different contexts, we have criticisms against the theory of comparative advantage. See Bhagwati, Panagariya, and Srinivasan (1998) and Parrinello (2006).
That is, the net output labour vector is one unit of English labour or that of Portuguese. Indeed, when unskilled labour in each country is not jointly produced by normal production processes, which is likely, we can prove a sort of nonsubstitution theorem under which our values do not change, even if the above vector $c$ takes on any positive values among unskilled labour of various countries with the remaining entries being zero. (See Fujimoto et al. (2003).)

The above sort of awkwardness does not appear when one country is not absolutely superior to the other in both industries. For example, when the coefficient $(B - A)_{44}$, for Portuguese wine production, increases from 3 to 3.01, we have

$$x^* \doteq (1.197, 0, 0, 1.590, 2.197, 2.590, 2.590, 0, 0, 0, 0, 2.197, 2.197)',$$

$$q^{**} \doteq (0.598, 0.795, 0.598, 0.795, 2.394, 2.394, 0.598, 0.795),$$

$$\lambda \doteq (0.25, 0.332, 0.25, 0.332, 0.582, 0.582, 0.25, 0.332).$$

showing complete specialization in each country and proper exchange in international markets. We can thus say that comparative or absolute advantage of a country over another in the production of a certain commodity is made clear after calculating values. When an awkward result comes out on the quantity side, i.e., in the principal problem (PG), it shows absolute advantage of a country over another.

At this point, the reader may have a question that the ratio of workers employed in two countries, here in this model 3.8 to 1.0, may not coincide with the actual ratio between two countries’ working populations, thus causing unemployment of simple labour in a country if we carry out the optimal solutions to the problem (PG) above. So, it seems we had better add one more constraint of a strict equality that the ratio of workers employed in two countries should be that of two countries’ working populations. When we add this constraint as 3 to 1 to the preceding numerical example, we get the following solutions.

$$x^* = (1.2, 1.4, 0, 0.2, 3.6, 1.2, 1.2, 1.2, 0.6, 0.6, 0, 0, 0, 0, 0)',$$

$$q^{**} = (0.6, 0.8, 0.6, 0.8, 2.4, 2.4, 0.6, 0.8),$$

$$\lambda \doteq (0.25, 0.333, 0.25, 0.333, 0.58333, 0.58333, 0.25, 0.333).$$

In this case, the values need not change, and can be obtained through the same problem (DG), with the shadow values to the added constraints set to zero. In general, the values change as the real ratio is far from that obtained through the problem (PG) without the constraint.

This additional equality constraint is, however, unnecessary simply because the problem (PG) has nothing to do with an efficiency seeking problem in the real world, as is noted in section 2. The problem (PL) is just auxiliary and virtual, and is useful only to understand the dual side better. When prices are
dealt with, the discrepancy from the actual working population ratio can mean uneven distribution of unemployment between two countries through international trade.

Now then, the problem is how to define prices for models with international trade. England will not be interested in international trade unless it brings forth profits calculated in terms of prices. And this task, we wish to perform with no resort to demand theory. Our method is to introduce instrumental efficiency conversion rates among unskilled labour of various countries. Thus, in the above model, we regard one unit of Portuguese labour as $(e_{<1})$ unit of English labour based on Portuguese inferiority in technical level of production. Again to make our story simpler, we assume the profit rate is zero, and $e = 0.5$. The programming problem is now

$$(DG) \quad \text{max } q' \cdot c \text{ subject to } q' B \leq q' \Lambda + b' \text{ and } q' \in R^8_+,$$

where $b' \equiv (0, 0, 0, 0, 1, e, 0, 0, 0, 0, 0, 0)$, and

$c \equiv (0, 0, 0, 0, 1, 1, 0, 0)$ or

$(0, 0, 0, 0, 1, 0, 0)$ or

$(0, 0, 0, 0, 1, 0, 0).$

$$(PG) \quad \text{max } b' \cdot x \text{ subject to } Bx \geq \Lambda x + c \text{ and } x \in R^{14}_+.$$
thought of as parameters reflecting relative political as well as military strengths of nations. When $e = 0$, Portugal becomes a perfect colony of England.

As the reader has noticed, our method can allow for any finite number of countries and of commodities. Besides, one can easily introduce transportation processes necessary to carry out international trade.

10. CONCLUDING REMARKS
To perceive alternative consumption baskets, one may be able to introduce utility functions as Johansen (1963) did, or to consider an array of diets which realize the necessary levels of nutritional elements and calories when consumed, which was also suggested in Johansen (1963). We do not need, however, utility functions, to incorporate alternative household activities. They can simply be what are observed in statistical surveys. The authors regard utility function as a subjective entity, thus as unreliable and far more changeable than various processes are changeable through technical progress. In one phrase, the price ratio determines the ratio between marginal utilities if utility functions should exist.

The principal linear programming (PGP) has nothing to do with reality even when the rates of profit are uniform among production processes. In the real world, growth or expansion/contraction rates of commodities are typically smaller than profit rates, causing a possible choice of inefficient processes. This was well discussed in Nuti (1970) by use of a neo-Austrian model. And yet, (PGP) is useful to make us notice that in the background of our determination of values and prices, an optimization in terms of physical units is latent.

A slightly weaker assumption than (AG) is that
Assumption (AGW): There is a scalar $\alpha$ such that $0 < \alpha < 1$, and a pair of matrices $B$ and $\alpha A$ satisfy the assumption (AG).
After calculating the values for $B$ and $\alpha A$ with $\alpha$ being a parameter, we let $\alpha$ approach to unity, and adopt the limiting values. In this way, we can consider the so low level of technology that it can at best realize stationary states as well.

Two remaining problems are how to deal with in a proper way services and exhaustible resources. Concerning the former, the reader is referred to Parrinello (2007).

REFERENCES

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The original German edition was published in 1867.


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