NON-LINEAR LEONTIEF MODELS IN ABSTRACT SPACES

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This paper is to elaborate, using a functional analysis approach, the mathematical aspects of scale-dependent input output models developed by Evans, Sandberg, Lahiri and Chander. All the basic results are established in abstract spaces of any dimension and in the case of existence theorems we do not require the continuity of the input function. A further generalization is made to a model with alternative processes. An application is also made to a model with indivisible commodities.

1. Introduction

Scale-dependent input–output models were first investigated by Evans (1954) and have been developed by Sandberg (1973), Lahiri (1976), Lahiri and Pyatt (1980), Fujimoto (1980a,b), Chien and Chan (1979) and Chander (1983) [see also Nikaido (1968, ch. III)]. When these models are cast into the framework of abstract functional analysis all the basic results still holds and the proofs remain simple as well. Among others, the existence of a solution can be established without requiring the continuity of the input function nor the convexity of the commodity space. We also discuss a model with alternative processes for each industry.

2. Abstract non-linear Leontief model

Let $X$ be a vector lattice over a subring $K$ of the real field $R$ and $X_+$ be the set $\{x|x \in X, x \geq 0\}$ [see, e.g., Yosida (1965, p. 364) for the definition of vector lattice over $R$: a vector lattice over a subring $K$ of $R$ can similarly be defined]. Let $T$ be a mapping from $X_+$ into itself with the property that $T0=0$. We deal with the equation

$$x = Tx + d,$$  \hspace{1cm} (1)

where $d$ is a given element in $X_+$. This corresponds to the equation $x = Ax + d$ in linear Leontief models, where $A$ is the $n$ by $n$ input coefficient matrix.

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and \( d \) the final demand \( n \) vector with \( X \) being \( R^n \), \( K \) being \( R \) and \( X^+ \) being \( R^+ \). In the case of scale-dependent models, the equation (1) corresponds to \( x = F(x)x + d \) in Lahiri (1976, p. 949). So we may call the mapping \( T \) the input function.

We consider the following assumptions:

A.1. The space \( X \) is order complete [see Yosida (1965, p. 367)].
A.2. The mapping \( T \) is isotone, i.e., \( Tx \leq Ty \) for \( x, y \in X^+ \), such that \( x \leq y \).
A.3. There exists an \( x^+ \in X^+ \) such that \( x^+ \geq Tx^+ + d \).

The assumption A.1 is satisfied, for instance, by the function space \( L^p \) as well as \( R^n \). The assumption A.2 means that more gross output asks more input in each commodity, and is weaker than Chander's Assumption I (1983, p. 221). It should be noted that we allow not only for variable returns to scale but also for externalities. As is noted in Fujimoto (1980a, p. 414), however, pure external economies are excluded, while external diseconomies are covered. Our assumption A.3 is a straightforward productiveness condition.

3. Existence of a solution

We can first show

**Theorem 1.** Given the assumptions A.1–A.3, there exists a solution \( x^* \in X^+ \) such that \( x^* = Tx^+ + d \).

**Proof.** Consider the order interval \([0, x^+] = \{ x \mid x \in X^+, 0 \leq x \leq x^+ \}\), where \( x^+ \) is an element given in A.3. Define a mapping \( F \) as

\[
F: x \in X^+ \to Tx + d.
\]

\( F \) is also isotone and maps the order interval \([0, x^+]\) into itself. Also define a set \( D = \{ x \mid x \in [0, x^+], x \geq Fx \} \). The set \( D \) is not empty by A.3. Now we can show \( x^* = \inf D \) is a solution to (1). By definition, \( x^* \leq x \) for each \( x \in D \). Since \( F \) is isotone \( Fx^* \leq Fx \leq x \) for each \( x \in D \), implying

\[
Fx^* \leq x^*.
\]

From (2), we have \( F(Fx^*) \leq Fx^* \), which means \( Fx^* \in D \), hence \( x^* = \inf D \leq Fx^* \). This inequality together with (2) shows that \( x^* = Fx^* \). Q.E.D.

This theorem is a simple application of Tarski's fixed point theorem [Tarski (1955, p. 286)]. Tarski's fixed point theorem was obtained in 1930's [see his explanation (1955, p. 286)]. Our theorem is also stated in Krasnoselskii (1964, Theorem 4.1 (a), p. 123) and he attributed it to Birkhoff (1948, p.54) [see Krasnoselskii (1964, p. 365)].
It should be noted that $X$ has no topology and $T$ has nothing to do with continuity, and that $K$ is a subring of $R$ with the identity element 1, and so $X$ need not be convex.

4. Uniqueness

We now make the fourth assumption

$A.4$. If $x \leq y$, $x \neq y$, then $x - Tx \neq y - Ty$.

This assumption is postulated in Sandberg (1975) and Yun (1979) in a stronger form for the case $X = R^n$. $A.4$ means that when the activity levels of some industries are raised, at least one commodity changes in its net output.

**Theorem 2.** *Given the assumptions $A.1$–$A.4$, there exists a unique solution to the equation $x = Tx + d$.*

**Proof.** If $y$ is a solution in $X_+$, then in particular $y \geq Fy$. Hence, as in the proof of Theorem 1, $x^* = \inf D$ is a solution whereby $D = \{x \mid x \in [0, y], x \geq Fx\}$. Since $x^* \leq y$ and $x^* - Tx^* = d = y - Ty$, the assumption $A.4$ yields $y = x^*$.

Q.E.D.

It is noted that the assumption $A.4$ is weaker than the requirement that $T$ is contractive when $X$ is a normed vector lattice [see also a uniqueness theorem due to Moré (1974, p. 329)].

5. Successive approximation

We can obtain an approximate solution to (1) by a simple iteration if $X$ is suitably topologized. So, we assume in place of the assumptions $A.1$ and $A.2$

$A.1^*$. $X$ is a reflexive Banach lattice over $R$ [Schaefer (1974, ch. II, §5)].

$A.2^*$. $T$ is isotone and continuous.

Again the function space $L^p(1 < p < \infty)$ satisfies $A.1^*$, to say nothing of $R^n$.

**Theorem 3.** *Given the assumptions $A.1^*$, $A.2^*$ and $A.3$, two sequences

$$
x^{k+1} = Tx^k + d \quad \text{with} \quad x^0 = 0, \quad k = 0, 1, 2, \ldots,
$$

$$
y^{k+1} = Ty^k + d \quad \text{with} \quad y^0 = x^k, \quad k = 0, 1, 2, \ldots,
$$

converge respectively to solutions of (1), $x^*$ and $y^*$.*
Proof. Clearly, \( x^0 \leq x^1 \leq x^2 \leq \cdots \leq y^2 \leq y^1 \leq y^0 \leq x^+ \). Thus, the two sequences \( \{x^k\} \) and \( \{y^k\} \) are both order bounded, and so norm bounded. Then by Theorem 5.11 in Schaefer (1974, p.91) both sequences converge in norm respectively to \( x^* \) and \( y^* \) [refer also to the remark below Corollary to Theorem 5.9 [Schaefer (1974, p. 89)]]). Since \( T \) is continuous by A.2*, we know \( x^* \) and \( y^* \) are solutions to (1). Q.E.D.

This is a variant of Kantorovich's result (1939) [see also Ortega and Rheinboldt (1970, p. 442)]. A similar result is also given by Krasnoselskii (1964, Theorem 4.1 (b), p. 123).

**Theorem 4.** Given the assumptions A.1*, A.2*, A.3 and A.4, the iterates

\[ z^{k+1} = Tz^k + d, \quad K = 0, 1, 2, \ldots, \]

starting from an arbitrary \( z^0 \in \{0, x^+\} \), converges to a unique solution.

**Proof.** First we show that the solution \( x^* \) in Theorem 3 above is the least solution among solutions to (1). If there is any other solution \( y \) such that \( y = Ty + d \), we have \( 0 \leq y \), and \( x^{k+1} = Tx^k + d \leq Ty + d = y \) for \( k = 0, 1, 2, \ldots \). Thus, \( x^* \leq y \). By the same argument in the proof of Theorem 2 above, there exists a unique solution to (1). Since \( 0 \leq z^0 \leq x^+ \), we have \( x^k \leq z^k \leq y^k \), where \( x^k \) and \( y^k \) are the iterates defined in Theorem 3. By Theorem 3, both \( \{x^k\} \) and \( \{y^k\} \) converge to the unique solution, and so does \( \{z^k\} \). Q.E.D.

This theorem is presented in Krasnoselskii (1964, p. 128), assuming the uniqueness of a solution.

6. Alternative processes

When each industry has a plural number of alternative processes, the input function \( T \) is not a point-to-point mapping, but a set-valued correspondence. Moreover, by the presence of externalities and variable returns to scale, we cannot expect that the image set \( Tx \) is convex or even contractible. This situation we will consider in an abstract setting. Let us state the assumptions we make below.

**A.1*.** \( X \) is a reflexive Banach lattice over \( R \).

**A.2***(i). \( T \) is isotone, i.e., if \( x \leq y \) for \( x, y \in X_+ \), then for any \( v \in Ty \) there exists a \( u \in Tx \) such that \( u \leq v \).

(ii). The image set \( Tx \) is compact for any \( x \in X_+ \).

**A.3***. There exists an \( x^t \in X_+ \) such that \( x^t \geq u + d \) for some \( u \in Tx^t \).

Needless to say, the mapping \( T \) is from \( x \in X_+ \) to a subset of \( X_+ \). We also assume \( T0 = \{0\} \).
Now we can prove

**Theorem 5.** Given the assumptions A.1*, A.2** and A.3**, there exists an \( x^* \) such that \( x^* = u^* + d \) for some \( u^* \in Tx^* \).

**Proof.** Define a set \( D \equiv \{ x | x \in X_+, x \geq u + d \text{ for some } u \in T x \} \). By A.3**, \( D \) is not empty. We first show that \( D \) is inductively ordered in the sense of the dual order, i.e., each totally ordered subset has a lower bound in \( D \). Take any totally ordered subset \( B \subset D \). By the latter half of Theorem 5.11 in Schaefer (1974, p. 91), the directed set \( B \) converges in norm to \( b^* \). We know \( h^* = \inf B \) by virtue of Lemma 5.8 [Schaefer (1974, p. 89)]. (Note here that the positive cone is closed in any topological vector lattice.) Now take any \( b \in B \). There is a \( c \in T b \) such that \( b \geq c + d \). By the fact that \( b^* \leq b \), we can find by A.2**(i) an element \( e \in T b^* \) such that \( e \leq c \). Thus we have \( b \geq e + d \) for some \( e \in T b^* \). This means that the set \( B \) is included in the set \( T b^* + d + X_+ \equiv \{ u + d + x | u \in T b^*, x \in X_+ \} \). Since this set \( T b^* + d + X_+ \) is closed by A.2**(ii), and the directed set \( B \) converges to \( h^* \), \( b^* \) is included in this set, implying \( b^* \geq u + d \) for some \( u \in T b^* \). Therefore, \( b^* \in D \). Clearly \( b^* \) is a lower bound of \( B \), and so \( D \) is shown to be inductively ordered.

Now by the Zorn's lemma, \( D \) has a minimal element \( x^* \). We have \( x^* \geq u^* + d \) for some \( u^* \in Tx^* \). From the inequality \( x^* \geq u^* + d \), it follows that there is a \( v \in T(u^* + d) \) such that \( u^* \geq v \). So, we have \( u^* + d \geq v + d \) for some \( v \in T(u^* + d) \), implying \( u^* + d \in D \). Since \( x^* \) is a minimal element and \( x^* \geq u^* + d \), the equation \( x^* = u^* + d \) must hold. Q.E.D.

It should be noted that the isotoneness is required 'collectively', and not for an individual technique which is implicit here. An extension of Tarski's fixed point theorem to set-valued mappings has been given in Fujimoto (1984), but there it is assumed directly that the image set \( Tx \) is inductively ordered.

7. Indivisible commodities

In our non-linear Leontief model presented in section 2, it is easy to handle indivisible commodities. Let \( Z \) and \( Z_+ \) be the set of integers and the set of non-negative integers respectively. Now suppose that there are \( n \) commodities, and for the sake of simplicity, every commodity is assumed indivisible. The minimum indivisible quantity of each commodity is taken as its unit of measurement, and any quantity of this commodity must be expressed as a non-negative integer. Thus, let our space \( X \equiv Z^* \equiv Z \times Z \times \cdots \times Z \) with an operating ring being \( Z \), and \( X_+ = Z_+ \). The order in \( X \) is the one introduced by the cone \( Z_+ \). The space \( Z^* \) satisfies the assumption A.1. Hence, if \( T \) satisfies the assumptions A.2 and A.3, then by Theorem 1 we know that there is a solution \( x^* \in Z^* \) such that \( x^* = T x^* + d \).
To prove this, we cannot use the Brouwer fixed point theorem because the order interval \([0, x^1]\) is not a convex set.

A further generalization is possible from a vector lattice over a subring to a lattice-ordered group. Beside Theorems 1 and 2, with some modification, Theorems 3 and 4 will also hold in this extension. For this, we change the assumptions as

\textit{A.1'}. \(X\) is a \(\sigma\)-complete lattice-ordered group.

\textit{A.2'}. \(T\) is isotone and \(\sigma\)-continuous.

The proofs in section 5 are still valid if we substitute order-convergence for norm convergence.

References


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