Shephard Duality Derived from Symmetric Duality in Homogeneous Programming

Takao Fujimoto*

Abstract This research note is to present an easy derivation of Shephard duality by use of symmetric duality in homogeneous programming. In this way, a new line of research topics emerges.

1 Introduction

This note presents a new method, depending upon an old result, of deriving Shephard duality. The present author proved in Fujimoto ([4]) a symmetric duality in homogeneous programming by use of a saddlepoint property by Uzawa ([8]). It seems that this new way is more natural and thus more digestible to economics students.

In Section 2, we explain symmetric duality in homogeneous programming, and in Section 3, Shepard duality is derived by use of this duality. In the final section, some suggestions for a new line of researches are given.

2 Symmetric Duality in Homogeneous Programming

Let us first explain our notation. The symbol $\mathbb{R}^n$ stands for the n-dimensional Euclidean space, and $\mathbb{R}^+_n$ for the nonnegative orthant.

*Faculty of Economics, Fukuoka University, Fukuoka, Japan
of $\mathbb{R}^n$. For vector comparison, $x \gg y$ for $x, y \in \mathbb{R}^n$ means $x_i > y_i$ for all $i$; $x > y$ means $x_i \geq y_i$ for all $i$ and $x \neq y$; $x \geq y$ means $x_i \geq y_i$ for all $i$. We use the same symbol both for a column vector and for its transpose. Thus, two vectors juxtaposed implies the inner product of them.

Eisenberg ([3]) extended the symmetric duality in linear programming to the following homogeneous programming problems:

$$(P) \inf \psi(x) \text{ subject to } x \in \mathbb{R}^n_{+} \text{ and } yH(x) \geq \tau(y) \text{ for all } y \in \mathbb{R}^m_{+};$$

$$(D) \sup \tau(y) \text{ subject to } y \in \mathbb{R}^m_{+} \text{ and } yH(x) \leq \psi(x) \text{ for all } x \in \mathbb{R}^n_{+},$$

where $\psi(x): \mathbb{R}^n_{+} \to \mathbb{R}$ and $\tau(y): \mathbb{R}^m_{+} \to \mathbb{R}$ are positively homogeneous of degree one, continuous, and convex and concave, respectively.

$$H(x) \equiv (H_1(x), \ldots, H_m(x))$$

is a linear mapping from $\mathbb{R}^n_{+}$ to $\mathbb{R}^m$.

Fujimoto ([4]) further generalized the above Eisenberg’s result to the case where $H(x)$ is not necessarily linear. Therein, the functions $\psi(x)$ and $\tau(y)$ are defined for given two nonempty convex sets $X \subseteq \mathbb{R}^n_{+}$ and $Y \subseteq \mathbb{R}^m_{+}$ as follows.

$$\psi(x) \equiv \sup \{zx \mid z \in X\}, \quad \tau(y) \equiv \inf \{zy \mid z \in Y\}.$$

The assumptions made in Fujimoto ([4]) are:

(A1) Each $H_i(x)$ is continuous, concave and positively homogeneous of degree one.

(A2) There exists an $x \in \mathbb{R}^n_{+}$ such that $H(x) \gg 0$, i.e., the Slater condition.

(A3) The set $X$ in $\mathbb{R}^n_{+}$ is bounded and there exists an $x \in X$ such that $x \gg 0$. 

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The symmetric duality theorem in Fujimoto ([4]) is:

**Theorem.** Given the assumptions (A1)-(A3), the problems (P) and (D) have the same value and have the optimal vectors.

### 3 Shephard Duality

Now we are ready to derive Shephard duality from our theorem in the previous section. First, we explain Shephard duality theorem. (See also Hackman ([5]) for the theorem and a simple geometric proof.) In our economy there is a production function \( F(z) \) from \( z \in \mathbb{R}^n_+ \) to an output vector \( q \in \mathbb{R}^k_+ \). The level set \( L(q) \) is defined as

\[
L(q) \equiv \{ z \in \mathbb{R}^n_+ \mid F(z) \geq q \}.
\]

The scalar cost function \( C(p, q) \) for \( p \in \mathbb{R}^n_+ \) and \( q \in \mathbb{R}^k_+ \) is then defined as

\[
C(p, q) \equiv \min_z \{ pz \mid z \in \mathbb{R}^n_+, z \in L(q) \}.
\]

Shephard ([7]) specified his distance function \( d(z, q) \) for \( z \in \mathbb{R}^n_+ \) and \( q \in \mathbb{R}^k_+ \) as

\[
d(z, q) \equiv \max_s \{ s \mid s \in \mathbb{R}_+, z/s \in L(q) \}.
\]

His duality theorem is expressed as

\[
d(z, q) \equiv \inf_p \{ pz \mid p \in \mathbb{R}^n_+, C(p, q) \geq 1 \}.
\]

Now let us fix \( z \) and \( q \), and regard \( C(p, q) \) as \( H(x) \) from \( \mathbb{R}^n_+ \) to \( R \) in the problem (P) in Section 2. (Note that the variable is \( p \).)
in place of $x$.) Then the problem
\[
\inf_p \{pz \mid p \in \mathbb{R}_+^n, \ C(p, q) \geq 1\}
\]
is rewritten in the form of (P)
\[
(P) \quad \inf \psi(p) \equiv pz \text{ subject to } p \in \mathbb{R}_+^n \text{ and } yC(p, q) \geq y \text{ for all } y \in \mathbb{R}_+.
\]
The problem (D) dual to this (P) is
\[
(D) \quad \sup \tau(y) \equiv y \text{ subject to } y \in \mathbb{R}_+ \text{ and } yC(p, q) \leq pz \text{ for all } p \in \mathbb{R}_+^n,
\]
which is equivalent to
\[
\max_y \{y \mid y \in \mathbb{R}_+, \ z/y \in L(q)\} = d(z, q).
\]
In these problems (P) and (D), both $\psi(p)$ and $\tau(y)$ are linear. By virtue of the symmetric duality theorem in the foregoing section, we have the desired Shephard duality.

4 New Line of Researches

Following van Moeseke ([6]), we can weaken the requirement of Slater condition (A2) as well as (A3). In so doing, we may also be able to generalize our result to symmetric duality in homothetic programming. That is, we first extend van Moeseke’s saddlepoint property to homothetic programming problem, and use it in place of Uzawa’s saddlepoint theorem ([8]). Generalizations to homothetic
programming enable us to deal with some sorts of variable returns to scale.

One more topic to be dealt with is directional distance functions introduced by Chambers, Chung and Färe ([1]). Some simple directional distance functions are certainly to be handled. The reader is referred also to Chavas and Cox ([2]).

Since the symmetric duality in Fujimoto ([4]) was proved while the author took into consideration a certain type of externalities, it seems possible to derive Shephard duality also for models with those properties.

We may also handle production correspondences rather than functions. All we need is the convexity of level sets.

References


