ASYMPTOTIC PROPERTIES FOR INHOMOGENEOUS ITERATIONS OF NONLINEAR OPERATORS

TAKAO FUJIMOTO† AND ULRICH KRAUSE‡

Abstract. The theorems on weak and strong ergodicity for inhomogeneous products of nonnegative matrices are extended to inhomogeneous iterations of nonlinear positive operators on Euclidean space. In particular some concave version of the Coale-Lopez theorem is presented and applied to a density-dependent Leslie model. The results are obtained, via Hilbert's projective pseudometric, from general theorems on inhomogeneous iterations of operators mapping a metric space into itself.

Keywords. discrete dynamical systems, weak and strong ergodicity, Hilbert's projective metric, nonlinear Leslie model

AMS(MOS) subject classifications. 47H05, 47H10, 54H20, 92A15

1. Introduction. Consider a discrete dynamical system given by an operator \( f \) mapping the state space into itself. For the dynamical behaviour of the system of particular interest are asymptotic properties of the iterates \( f^n \) if \( n \to \infty \). If, however, the system itself changes in the course of time with \( f_t \) as "law of motion" at, say, time \( t \), then one will become interested in asymptotic properties of the inhomogeneous iterations \( f_n \cdot f_{n-1} \cdot \cdots \cdot f_1 \) if \( n \to \infty \). This kind of problem arises in mathematical biology and mathematical economics where, e.g., the principle governing the growth of a population or the choice of a technology itself depends on time (cf. [4], [7], [8], [13], [14]). For the case of linear operators in finite dimensions, inhomogeneous iterations become inhomogeneous products \( A_n A_{n-1} \cdots A_1 \) of matrices, the asymptotic properties of which have been investigated for a long time in the theory of Markov chains and the theory of nonnegative matrices, respectively (cf. [14]). Since absolute magnitudes tend thereby to grow exponentially, one considers relative magnitudes as exemplified by \( x_n = A_n A_{n-1} \cdots A_1 x/\|A_n A_{n-1} \cdots A_1 x\| \), \( x \) being a starting vector and \( \| \cdot \| \) some vector space norm. There are two major stability results here, one on so-called strong ergodicity, meaning convergence of \( x_n \) for arbitrary starting vectors to the same limit and the other one on so-called weak ergodicity, meaning convergence of \( x_n - y_n \) to 0 for any two starting vectors \( x, y \) and \( y_n \) starting with \( y \). These results have important applications, in particular to population dynamics, where the second result is known also as the Coale-Lopez theorem (cf. [4], [8], [13], [14]).

As in other fields too, linearity is a strong idealization concerning applications and on behalf of the latter results on nonlinear operators, e.g., concave ones, are requested. We present in § 3 as the main results of this paper theorems on weak and strong ergodicity for positive nonlinear operators in finite dimensions that contain the well-known theorems on nonnegative matrices as special cases. In particular, a concave version of the Coale-Lopez theorem is presented. Section 4 provides some concrete classes of nonlinear examples for these results by developing a density- and time-dependent version of the Leslie model of population dynamics.

To prove the theorems of § 3 we translate the order-theoretic framework of positive operators into a metric framework and prove in § 2 the corresponding theorems. The metric used is Hilbert's projective pseudometric which was first introduced by

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* Received by the editors September 22, 1986, accepted for publication (in revised form) August 4, 1987.
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Birkhoff [1, 2] into functional analysis. He used it to extend Jentzsch’s theorem on linear integral operators to abstract linear operators by applying Banach’s contraction mapping principle with respect to this metric (cf. also [10], [11]). Hilbert’s metric is applied to inhomogeneous products of matrices in [8] and in [14, 2nd ed.] (cf. also [4]). In extending Birkhoff’s theorem to nonlinear operators, Hilbert’s metric is used in [11] within an infinite-dimensional nonlinear context (cf. also [12] for the finite-dimensional case). The present paper applies Hilbert’s metric to inhomogeneous iterations of nonlinear operators in finite dimensions. (A direct approach to inhomogeneous iterations in finite dimensions that does not involve Hilbert’s metric, but that is by no means more simple, can be found for matrices in [14, 1st ed.] and for nonlinear operators in [6].) In the case of homogeneous iterations usually one of the many variations of the contraction mapping principle is applied with respect to Hilbert’s metric. To treat inhomogeneous iterations, however, something different is needed. Although there is an enormous literature on the contraction mapping principle, as surveyed, e.g., in [3], [5], and [10], there seems to be none handling the composition of several different contractions. Hence in §2 we give a systematic account of inhomogeneous iterations within the metric framework. Although in the present paper the material of §2 is used in §3 only for the finite-dimensional case, it may also be applied to infinite dimensions.

2. Inhomogeneous iterations of operators mapping a metric space into itself. An operator \( f : X \to X \) on a metric space \( X \) with metric \( d \) is said to be nonexpansive if \( d(f(x), f(y)) \leq d(x, y) \) for all \( x, y \in X \); it is said contractive on \( Y \), for \( Y \subseteq X \), if

\[
d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \in Y \text{ with } x \neq y.
\]

We call a sequence \((f_n)\) of operators \( f_n : X \to X \) an (asymptotically) contractive sequence on \( Y \) for \( Y \subseteq X \), if there exists a continuous mapping \( c : Y \times Y \to \mathbb{R} \) such that the following two conditions are satisfied:

(i) \( c(x, y) < d(x, y) \) for all \( x, y \in Y \) with \( x \neq y \);

(ii) To every \( \varepsilon > 0 \) there exists a \( N(\varepsilon) \in \mathbb{N} \) such that \( d(f_n(x), f_n(y)) \leq c(x, y) + \varepsilon \) for all \( n \geq N(\varepsilon) \), all \( x, y \in Y \).

For a given \( r \geq 1 \) and a given sequence \((f_n)\) of operators on \( X \) we will consider also the sequence of lumped operators \((F_m)\) defined by

\[
F_m = f_m \circ f_{m+1} \circ \cdots \circ f_1
\]

(where \( \circ \) stands for the composition of mappings).

In what follows we are concerned with the asymptotic behaviour of inhomogeneous iterations, which means the behaviour of \( f_n \cdots f_2 \cdot f_1(x) \) for \( n \to \infty \) where \( x \in X \) and \((f_n)\) is a sequence of operators on \( X \). In the special case of (homogeneous) iteration the underlying sequence is simply \((f, f, \cdots)\) for some operator \( f \) on \( X \). This is a contractive sequence precisely when \( f \) is a contractive operator, and the sequence of lumped operators in this case is \((f', f', \cdots)\).

The following theorem provides conditions under which inhomogeneous iterations come close together irrespective of the starting point. This does not necessarily mean that the iterations itself do converge. (In the next section we will see that the former is related to so-called weak ergodicity and the latter to strong ergodicity.)

**Theorem 1.** Let \((f_n)\) be a sequence of nonexpansive operators on the metric space \((X, d)\) such that for some \( r \geq 1 \) the sequence \((F_m)\) of lumped operators is contractive on \( Y \) and satisfies \( F_m(X) \subseteq Y \) for some compact subset \( Y \) of \( X \) and almost all \( m \). Then

\[
\lim_{n \to \infty} d(x_n, y_n) = 0 \quad \text{for any two sequences defined by} \quad x_{n+1} = f_n(x_n) \quad \text{and} \quad y_{n+1} = f_n(y_n)
\]

with arbitrary starting points \( x_1, y_1 \in X \).
Proof. (1) Consider first a sequence \((g_m)\) of nonexpansive operators, contractive on \(Y\) and satisfying \(g_m(X) \subseteq Y\) for some compact \(Y \subseteq X\) and almost all \(m\). Let \(x_{m+1} = g_m(x_m), y_{m+1} = g_m(y_m)\) for \(m \in \mathbb{N}\) and \(x_1, y_1 \in X\) arbitrary. Since eventually \((x_m, y_m) \in Y \times Y\) and \(Y\) is compact, there exists a subsequence \((x_{k(m)}, y_{k(m)})\) converging to some \((x^*, y^*) \in Y \times Y\). By the nonexpansiveness of \(g_m\)
\[
d(x_{m+1}, y_{m+1}) = d(g_m(x_m), g_m(y_m)) \leq d(x_m, y_m),
\]
and hence \(\lim_{m \to \infty} d(x_m, y_m) = d(x^*, y^*)\).

The sequence \((g_m)\) is also contractive and according to the definition there exists a function \(c\) and to every \(\varepsilon > 0\) an \(M(\varepsilon)\) such that
\[
d(x_{k(m)+1}, y_{k(m)+1}) = d(g_{k(m)}(x_{k(m)}), g_{k(m)}(y_{k(m)})) \leq c(x_{k(m)}, y_{k(m)}) + \varepsilon
\]
for all \(m \geq M(\varepsilon)\). Letting \(m \to \infty\) from this we obtain \(d(x^*, y^*) \leq c(x^*, y^*) + \varepsilon\) because of \(d(x^*, y^*) \leq d(x_m, y_m)\) and the continuity of \(c\). Since \(\varepsilon > 0\) was arbitrary \(d(x^*, y^*) \leq c(x^*, y^*)\) which together with \(c(x, y) < d(x, y)\) for \(x \neq y\) and \(x, y \in Y\) yields \(x^* = y^*\). Thus finally \(\lim_{m \to \infty} d(x_m, y_m) = 0\).

(2) Suppose now \((f_n)\) is a sequence as in the theorem and put \(g_m = F_{(m-1)r+1}\). Being a composition of nonexpansive mappings, \(g_m\) is nonexpansive. Step (1) therefore yields \(\lim_{m \to \infty} d(x_m, y_m) = 0\) for sequences defined by \(x_{m+1} = g_m(x_m), y_{m+1} = g_m(y_m), x_1 = x_1, y_1 = y_1\). By the definition of lumped operators
\[
g_m = g_m \cdot g_{m-1} \cdot \cdots \cdot g_1 \cdot g_{m-r} \cdot g_{m-r-1} \cdots \cdot f_2 \cdot f_1,
\]
and hence \(x_{m+1} = x_{m+r+1}, y_{m+1} = y_{m+1}\). For any natural number \(n\) there exist nonnegative integers \(m(n)\) and \(i\) such that \(n = m(n)r+i\) with \(0 \leq i < r\). Since \(x_{n+i} = f_n \cdot f_{n-1} \cdots f_{n-r+1}(x_{n-r+1})\) and the \(f_n\)'s are nonexpansive it follows that
\[
d(x_{n+i}, y_{n+i}) \leq d(x_{m(n)r+i}, y_{m(n)r+i}) = d(x_{m(n)+1}, y_{m(n)+1}).
\]
Thus \(\lim_{n \to \infty} d(x_n, y_n) = 0\).

Remark. The above proof shows that Theorem 1's conclusion remains valid if instead of the \(F_m\)'s the operators \(g_m = F_{(m-1)r+1}\) are considered with \(g_m(X) \subseteq Y\) for some compact \(Y \subseteq X\) and almost all \(m\) and such that the sequence \((g_m)\) contains a contractive subsequence on \(Y\).

Now we are looking for conditions ensuring the convergence of the inhomogeneous iterations itself to a common limit for arbitrary starting points. Obviously this is stronger than the convergence statement made in Theorem 1 and therefore we have to add some assumptions.

THEOREM 2. Let \((f_n)\) and \((F_n)\) be sequences of operators as in Theorem 1. Suppose in addition for the lumped operators \(F_n\) uniform convergence on the metric space to some operator \(F\). This assumption is particularly fulfilled in case the operators \(f_n\) converge uniformly to some \(f\). Then for arbitrary starting points \(x_1 \in X\) the sequence defined by \(x_{n+1} = f_n(x_n)\) converges to the unique fixed point of \(F\), or \(f\), respectively.

Proof. (1) Consider first a sequence \((g_m)\) of nonexpansive operators, contractive on \(Y\) and satisfying \(g_m(X) \subseteq Y\) for some compact \(Y \subseteq X\) and almost all \(m\). Assume \(g_m\) converges uniformly on \((X, d)\) to some operator \(g\), i.e., to \(e > 0\) there exists \(N_1(e)\) such that \(d(g_m(x), g(x)) \leq e\) for all \(m \geq N_1(e)\) and all \(x \in X\). Since \((g_m)\) is contractive on \(Y\) there exists a function \(c\) and to \(e > 0\) there exists \(N_2(e)\) such that
\[
d(g_m(x), g_m(y)) \leq c(x, y) + e\quad \text{for all} \quad m \geq N_2(e)\quad \text{and all} \quad x, y \in Y.
\]

Therefore
\[
d(g(x), g(y)) \leq d(g(x), g_m(x)) + d(g_m(x), g_m(y)) + d(g_m(y), g(y)) \leq c(x, y) + 3e\quad \text{for} \quad m \geq N_2(e), m \geq N_2(e).
\]
This yields $d(g(x), g(y)) \leq c(x, y) < d(x, y)$ for $x \neq y$, $x, y \in Y$, i.e., $g$ is contractive on $Y$.

Let now $x_i \in X$ be an arbitrary starting point, fixed in what follows. We want to show that the set $A$ of all limit points of the set $\{x_m\}$, where $x_{m+1} = g_m(x_m)$, consists precisely of the unique fixed point of $g$. Since $(x_m)_m$ is eventually contained in the compact set $Y$, $A \subseteq Y$ and $A \neq \emptyset$. Pick some $x \in A$. Then there is a subsequence $(x_{j(n)})_n$ of $(x_m)_m$ converging to $x$ and we may assume that $x_{j(n)-1} \in Y$ for all $n$. $(x_{j(n)-1})_n$ contains a subsequence $(x_{k(n)-1})_n$ converging to some $y \in Y$. Obviously

$$d(x, g(y)) \leq d(x, x_{k(n)}) + d(\delta y_{k(n)-1} x_{k(n)-1}), g(x_{k(n)-1})) + d(g(x_{k(n)-1}), g(y))$$

and taking into account that $(x_{k(n)})_n$ converges to $x$, $(g_m)_m$ converges uniformly to $g$ and that $g$ is contractive on $Y$ we obtain $d(x, g(y)) = 0$, i.e., $x = g(y)$. Because of $y \in A$ by the same argument we can find $y_2 \in A$ such that $y = g(y_2)$. By iteration we obtain to every $n \in \mathbb{N}$ a $y_n \in A$ such that $x = g^n(y_n)$. Since $y_n \in A \subseteq Y$, $(y_n)_n$ contains a subsequence $(y_{h(n)})_n$ converging to some $y^* \in Y$. Because of $g(y) \in Y$, $g$ is contractive also on the metric space $(Y, d)$ and hence by a well-known version of Banach's contraction mapping principle $(g^n(y^*))_n$ converges to the unique fixed point $x^*$ of $g$. From $d(x, x^*) \equiv d(g(h(n)(y_{h(n)})) = g^h(n)(y^*) + d(g^h(n)(y^*), x^*)$ we therefore obtain $d(x, x^*) = 0$, i.e., $x = x^*$. This proves $A = \{x^*\}$, i.e., the convergence of the sequence $(x_m)_m$, defined by $x_{m+1} = g_m(x_m)$ with arbitrary starting point, to the unique fixed point of $g$.

(2) Let for arbitrary $x_i \in X$ $(x_i)_i$ be defined by $x_{i+1} = f_i(x_i)$. We fix $i$ with $0 \leq i < r$ and define $g_m = F_{m-r+1}$. By the assumptions made on the lumped operators $F_m$ we may apply step (1) to the sequence $(g_m)_m$. Therefore the sequence defined by $x_{m+1} = g_m(x_m)$, $x_i = x_{i+1}$ converges to the unique fixed point $x^*$ of $g = F$.

By the definition of lumped operators

$$g_m \cdot g_{m-1} \cdots \cdot g_1 = f_{m-1} \cdot f_{m-r+i-1} \cdots f_{i+1},$$

and hence $x_{m+1} = x_m + 1$. Therefore for any fixed $0 \leq i < r$ $(x_{m+r+i})_m$ converges to the unique fixed point $x^*$ of $F$. Since any natural $n$ can be written as $n = nr + i$ with $0 \leq i < r$ it follows that $(x_n)_n$ converges to $x^*$. This proves the theorem in case the lumped operators converge uniformly to some $F$.

(3) Let $(f_i)_n$ be an arbitrary sequence of operators on a metric space $X$ converging uniformly on $X$ to some uniformly continuous operator $f$. We first show that to every $\varepsilon > 0$ and to every natural $k$ there exists an $N(\varepsilon, k)$ such that

$$(*) \quad d(f^k(x), f_{n_1} \cdots f_{n_k}(x)) \equiv \varepsilon \quad \text{for} \quad n_i \equiv N(\varepsilon, k) \quad \text{and all} \quad x \in X.$$

By assumption, for $\varepsilon > 0$ given there exist $\delta(\varepsilon) > 0$ and $N(\varepsilon)$ such that

$$d(f(x), f_n(y)) = d(f(x), f(y)) + d(f(y), f_n(y)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

provided $d(x, y) \leq \delta(\varepsilon)$ and $n \equiv N(\varepsilon)$.

Hence, $(*)$ holds for $k = 1$ with $N(\varepsilon, 1) = N(\varepsilon)$. Suppose, $(*)$ holds for some $k \equiv 1$.

Then

$$d(f^k(x), f_{n_1} \cdots f_{n_k}(x)) = \delta(\varepsilon) \quad \text{for all} \quad n_1, \cdots, n_k, \equiv N(\varepsilon, k).$$

Putting $N(\varepsilon, k+1) = \max \{N(\varepsilon), N(\delta(\varepsilon), k)\}$, we obtain

$$d(f^{k+1}(x), f_{n_1} \cdots f_{n_k}(x)) = d(f(f^k(x)), f_{n_1} \cdots f_{n_k}(x)) \equiv \varepsilon$$

for $n_1, n_2, \cdots, n_{k+1} \equiv N(\varepsilon, k+1)$. This proves $(*)$ to hold for all $k$. 
Now, let \((f_n)_n\) as in Theorem 2 and Theorem 1, respectively, and suppose uniform convergence to some operator \(f\). Since every \(f_n\) is nonexpansive, \(f\) must be uniformly continuous. Thus \((\ast)\) applies and yields for \(e > 0\) given \(d(f(x), F_m(x)) \leq e\) for all \(m \geq N(e)\) and all \(x \in X\). Therefore \((F_m)_m\) converges uniformly on \(X\) to \(F = f\). Step (2) then yields convergence of the sequence defined by \(x_{n+1} = f_n(x_n), x_i \in X\), to the unique fixed point \(x^*\) of \(F\). By nonexpansiveness of \(f_n\) and uniform convergence to \(f\), from

\[
d(f(x^*), x^*) \leq d(f(x^*), f(x_n)) + d(f(x_n), f_n(x_n)) + d(x_{n+1}, x^*)
\]

it follows that \(f(x^*) = x^*\). □

**Remark.** Theorem 2 remains valid, if instead of the whole sequence \((F_m)_m\) only some subsequence is required to be contractive on \(Y\). The beginning of step (1) in the proof, when applied to this subsequence yields that \(F\) is contractive on \(Y\). But then \((F_m)_m\) is contractive on \(Y\) too. For, it is true in general that a sequence of operators \(h_m\) converging uniformly to some contractive \(h\) must be contractive (with \(c(x, y) = d(h(x), h(y))\)).

From Theorem 2 we may derive the following criterion which is cast more directly in terms of the given operators.

**Theorem 3.** For any sequence \((f_n)_n\) of nonexpansive operators on the metric space \(X\) the sequence defined by \(x_{n+1} = f_n(x_n)\) converges for arbitrary \(x_1 \in X\) to the same limit point, provided the following condition is satisfied: For some \(r \geq 1\) the sequence \((F_m)_m\) of lumped operators converges uniformly on \(X\) to some operator \(F\) for which there exists an open and relatively compact subset \(U\) of \(X\) such that \(F(X) \subseteq U\) and \(F\) is contractive on the closure \(\bar{U}\).

This condition is particularly satisfied if \((f_n)_n\) converges uniformly on \(X\) to some \(f\) for which \(f'(X) \subseteq U\) and \(f'\) is contractive on \(\bar{U}\) for some \(r \geq 1\), \(U\) being an open and relatively compact subset of \(X\).

**Proof.** To derive the above criterion from Theorem 2 we only have to show that \((F_m)_m\) is contractive on \(Y\) and for almost all \(m\) \(F_m(X) \subseteq Y\) for some compact subset \(Y\) of \(X\). Putting \(Y = \bar{U} F\) is contractive on \(Y\) and hence (cf. the remark following the proof of Theorem 2) the sequence \((F_m)_m\) is contractive on \(Y\). It remains to show that \(F_m(X) \subseteq Y\) for almost all \(m\). To every \(x \in X\) there exists some \(\varepsilon(x) > 0\) such that \(B(F(x), \varepsilon(x)) \subseteq U, B(F(x), \varepsilon(x))\) being the open ball with center \(F(x)\) and radius \(\varepsilon(x)\). Obviously the closure \(\bar{F}(X)\) is contained in \(\bigcup_{x \in X} B(F(x), \frac{1}{4} \varepsilon(x))\) and there is a finite cover, \(\bar{F}(X) \subseteq \bigcup_{x \in \bar{X}} B(F(x), \frac{1}{4} \varepsilon(x))\) for some finite set \(\bar{X} \subseteq X\), because \(F(X)\) is contained in the compact set \(\bar{U}\). Let \(\varepsilon\) be the smallest of the numbers \(\frac{1}{4} \varepsilon(x), x \in \bar{X}\). By uniform convergence of the lumped operators there exists \(N(\varepsilon)\) such that \(d(F_m(x), F(x)) \leq \varepsilon\) for all \(m \geq N(\varepsilon)\) and all \(x \in X\). For \(x \in X\) there exists some \(\hat{x} \in \bar{X}\) such that \(F(x) \in B(F(\hat{x}), \frac{1}{2} \varepsilon(\hat{x}))\) and therefore

\[
d(F_m(x), F(\hat{x})) \leq d(F_m(x), F(x)) + d(F(x), F(\hat{x})) \leq \frac{1}{2} \varepsilon(\hat{x}) + \frac{1}{2} \varepsilon(\hat{x}) = \varepsilon(\hat{x})
\]

for all \(m \geq N(\varepsilon)\). Hence \(F_m(x) \in B(F(\hat{x}), \varepsilon(\hat{x})) \subseteq U\) for all \(x \in X\), all \(m \geq N(\varepsilon)\). Thus \(F_m(X) \subseteq Y\) for almost all \(m\). □

**Remark.** In the situation of Theorem 3 the meaning of being contractive for the sequence of lumped operators is that some iterate of the limit function of the original sequence is contractive. More precisely, let \((f_n)_n\) be a sequence of nonexpansive operators converging uniformly on a metric space to some operator \(f\). Then for any \(r \geq 1\), the sequence \((F_m)_m\) of lumped operators is contractive if and only if \(f'^r\) is contractive. This is immediate by parts (1) and (3) in the proof of Theorem 2 and the remark thereafter.
3. Inhomogeneous iterations of nonlinear positive operators on Euclidean space. The results of the previous section we shall now apply to obtain the theorems of weak and strong ergodicity for inhomogeneous iterations of nonlinear positive operators on Euclidean space. As in the case of linear operators, these theorems have applications in mathematical biology. We shall obtain a concave version of the Coale–Lopez theorem of population dynamics as a corollary, which then is applied in the next section to a density-dependent Leslie model.

Let $E$ denote the $k$-dimensional Euclidean space with typical element $x = (x_1, \ldots, x_k)$, $x_i \in \mathbb{R}$. For $x, y \in E$ we write $x \leq y$ if $x_i \leq y_i$ for all $i$; we write $x < y$ if $x_i < y_i$ for all $i$. $E_+$ denotes the positive cone $E_+ = \{x \in E \mid x \geq 0\}$. By a scale we mean a continuous functional $p : E_+ \rightarrow \mathbb{R}_+$ with $p(0) = 0$ only for $x = 0$ and such that $p$ is positively homogeneous, i.e., $p(\lambda x) = \lambda p(x)$ for $x \in E_+$, $\lambda \in \mathbb{R}_+$, and $p$ is monotonic, i.e., $p(x) \leq p(y)$ for $0 \leq x \leq y$. Obviously every monotonic norm is a scale, but there are others, e.g., maxima or minima of these norms. In what follows we fix on $E_+$ an arbitrary scale $p$ and we denote its unit level set by $X$, $X = \{x \in E_+ \mid p(x) = 1\}$. This set $X$ when equipped with a metric defined below will serve as the metric space underlying the previous section. As for the operators we shall employ various properties in the sequel. An operator $T : E_+ \rightarrow E_+$ is

- proper if $Tx = 0$ is equivalent to $x = 0$;
- subhomogeneous if for $x, y \in X$, $0 \leq \lambda \leq 1$, $\lambda x \leq y$ implies $\lambda Tx \leq Ty$;
- ray-preserving if for every $x \in X$ and $\lambda > 0$ there exists some $\lambda' > 0$ such that $T(\lambda x) = \lambda' Tx$;
- ascending (for $p$; cf. [11]) if there exists a continuous mapping $\varphi$ of the unit interval $[0, 1]$ into itself with $\varphi < \varphi(\lambda)$ for $0 < \lambda < 1$ and such that for any $\lambda \in [0, 1]$ and any $x, y \in E_+$, $\lambda x \leq y$ implies $\varphi(\lambda) Tx \leq Ty$;
- pointwise bounded (for $p$) if for every $x \in X$ there exist $u(x), v(x) \in E_+$, $u(x) > 0$, such that $u(x) \leq Tx \leq v(x)$.

A sequence $(T_n)_n$ of operators $T_n : E_+ \rightarrow E_+$ is uniformly ascending, if all operators are ascending with the same $\varphi$, i.e., $\lambda x \leq y$ implies $\varphi(\lambda) T_n x \leq T_n y$ for all $n$. Similarly a sequence is uniformly pointwise bounded if $u(x) \leq T_n x \leq v(x)$ for all $n$.

On the unit level set $X$ of the scale we now consider Hilbert’s projective pseudometric, or Hilbert’s metric for short (cf. [1], [2], [4], [8], [10], [11], [12], [14]). For $x, y \in E_+ \setminus \{0\}$ let $\lambda(x, y) = \sup \{\lambda \in \mathbb{R}_+ \mid \lambda x \leq y\}$ and let $\mu(x, y) = -\log [\lambda(x, y) \cdot \lambda(y, x)]$. It is easily verified that $\mu$ is a metric on $X$ except that $\mu$ may take on the value $+\infty$. The following lemma translates properties of positive operators into properties of operators acting on the metric space $(X, \mu)$.

**Lemma 1.** For an operator $T : E_+ \rightarrow E_+$ which is proper let $\tilde{T} : X \rightarrow X$, $\tilde{T} = T/\mu(Tx)$ for $x \in X$.

(i) $T$ subhomogeneous $\Rightarrow \tilde{T}$ nonexpansive (on $(X, \mu)$).

(ii) $S$ proper and ray-preserving $\Rightarrow S \cdot \tilde{T} = \tilde{T} \cdot S$.

(iii) $T$ ascending $\Rightarrow \mu(\tilde{T}x, \tilde{T}y) \leq c(x, y)$ for all $x, y \in X$. Thereby $c(x, y) = -\log [\varphi(\lambda(x, y)) \cdot \varphi(\lambda(y, x))]$ ($\varphi$ as in the definition of "ascending") is continuous on $\{(x, y) \in X \times X \mid x > 0, y > 0\}$ with respect to Euclidean topology. Furthermore, $c(x, y) \leq \mu(x, y)$ for all $x, y \in X$ and $c(x, y) < \mu(x, y)$ if $x \neq y$ and $\mu(x, y) < +\infty$.

(iv) $T$ pointwise bounded and subhomogeneous $\Rightarrow$ There exists $a, b \in X$, $a \leq \tilde{T}x \leq b$ for all $x \in X$. If $(T_n)_n$ is uniformly pointwise bounded, then the bounds $a, b$ are independent of $n$.

**Proof:** (i) Since $T$ is subhomogeneous it follows that $\lambda(\tilde{T}x, \tilde{T}y) \geq (p(Tx)/p(Ty))\lambda(x, y)$ for all $x, y \in X$. Hence by the definition of Hilbert's metric $\mu(\tilde{T}x, \tilde{T}y) \leq \mu(x, y)$, i.e., $\tilde{T}$ is nonexpansive on $(X, \mu)$. 

(ii) With $S$, $T$ proper, $S \cdot T$ is proper too. Since $S$ is ray-preserving, for a given $x \in X$ there exists $\lambda^* > 0$ such that $S(Tx/p(Tx)) = \lambda^* S(Tx)$. Hence

$$\tilde{S}(\tilde{Tx}) = \frac{S(\tilde{Tx})}{p(S(\tilde{Tx}))} = \frac{\lambda S \cdot T(x)}{p(\lambda S \cdot T(x))} = \frac{S \cdot T(x)}{p(S \cdot T(x))} = \tilde{S}(T(x)).$$

(iii) Applying $p$ to $\lambda x \leq y$ it follows that $\lambda(x, y) \in [0, 1]$ for $x, y \in X$. Since $T$ is ascending it follows that $\varphi(\lambda(x, y)) \leq T y$, and hence $\varphi(\tilde{S}(\tilde{Tx}, \tilde{Ty})) \in p(Tx)/p(Ty) \varphi(\lambda(x, y))$.

Interchanging the roles of $x$ and $y$ the definition of $\mu$ yields $\mu(\tilde{Tx}, \tilde{Ty}) \leq c(x, y)$ with $c(x, y)$ as stated in the assertion. Suppose $x, y \in X$, $x \neq y$, and $\mu(x, y) < +\infty$. Because of the latter, $0 < \lambda(x, y)$ and $0 < \lambda(y, x)$, $\lambda(x, y) \leq 1$ and $\lambda(y, x) \leq 1$, because of $x, y \in X$, and equality in both cases would imply $x \leq y$ and $y \leq x$, i.e., $x = y$. Without restriction we may assume $\lambda(x, y) < 1$. Therefore by the properties of $\lambda(x, y) \cdot \lambda(y, x) < \varphi(\lambda(x, y)) \cdot \varphi(\lambda(y, x))$, implying $c(x, y) < \mu(x, y)$. From this $c(x, y) \leq \mu(x, y)$ for $x, y \in X$. Finally, $c(x, y)$ depends continuously on $x > 0, y > 0$ since by an easy calculation $\lambda(x, y) = \min \{ y_i/x_i, i \in \{1, \ldots, k\} \}$.

(iv) We shall show $u \leq Tx \leq v$ with $u, v \in E_+$, $u > 0$. Then $p(u) \leq p(Tx) \leq p(v)$, and (iv) follows by setting $a = u/p(v)$, $b = v/p(u)$. Let $T$ be subhomogeneous and $u(x) \leq Tx \leq v(x)$ for $x \in X$, $u(x), v(x) \in E_+$, $u(x) > 0$. Denote by $e_i, i = 1, \ldots, k$, the vector in $E = R^k$ having 1 in component $i$ and 0's otherwise and let $e = e_1 + \cdots + e_k$. Define $u = (\min_i p(e_i)/p(e)) \min_i u(e_i/p(e_i))$, where $\min_i u(e_i/p(e_i))$ is a vector the $i$th component of which is obtained by taking the minimum over $i$ of the $i$th component of $u(e_i/p(e_i))$. Define $v = (p(e)/\min_i p(e_i)) \min_i v(e_i/p(e_i))$. Obviously, $u, v \in E_+$ and $u > 0$. To see $u \leq Tx \leq v$, let $x \in X$. If $x_i$ denotes the $i$th component of $x$, then $x_i e_i \leq x \leq (x_i, x)$ and by applying $p \cdot p(e_i) \leq p(x) = 1 \leq (\max_i x_i) \varphi(e_i)$ and hence $\varphi(x_i) \varphi(e_i) \varphi(x_i) \leq \max_i p(e_i)$ since $T$ is subhomogeneous, from $(\min_i p(e_i))/p(e_i) \leq x_i$ it follows that $(\min_i p(e_i)/p(e)) \leq x_i$ subhomogeneity of $T$ yields $x_i p(e_i) \leq T(e_i/p(e_i)) \leq x$.

Hence $(\max_i x_i) p(e) u \leq Tx$ and $u \leq Tx$ because of $(\max_i x_i) p(e) \leq 1$. Finally, the independence statement holds by construction of $a$ and $b$.

To apply the results of the previous section, we need the following comparison of Hilbert's metric and the maximum metric $|x-y| = \max_i |x_i-y_i|$ as defined on $R^k$. $(x_i$ the $i$th component of $x)$

**Lemma 2.** For any $x, y \in X$

$$(\min_i x_i) [1 - \exp(-\mu(x, y))] \leq |x-y| \leq \max_i \{x_i, y_i\} (1 - \exp(-\mu(x, y))).$$

**Proof.** To see the first inequality, let $r = \min_i x_i, \Gamma r \leq |x-y|$, then the first inequality holds trivially. Suppose $r > |x-y|$. Obviously $r(x-y) \leq |x-y|$, and hence $1 - (|x-y|/r) x \leq y$. Thus $\lambda(x, y) \leq 1 - (|x-y|/r)$ and $\lambda(x, y) \cdot \lambda(y, x) \geq (1 - (|x-y|/r))^2$. Because of $\lambda(x, y) \cdot \lambda(y, x) = \exp(-\mu(x, y))$ this proves the first inequality. For the second inequality observe that $x-y \leq (1 - \lambda(x, y)) x \leq (1 - \lambda(x)) \lambda(y, x) x$ and therefore $x-y \leq (1 - \lambda(x, y) \lambda(y, x)) x$. By interchanging the roles of $x$ and $y$ and taking the maximum over $i$ the second inequality is obtained.

**Remark.** The proof shows that the first inequality is true for any $x, y \in E_+ \setminus \{0\}$.

Our first application is concerned with weak ergodicity, a concept which originally stems from the theory of Markov chains (cf. [4], [8], [13], [14]). More generally, there
holds weak ergodicity (relative to some fixed scale $p$) for a sequence $(T_n)_n$ of operators on $\mathbb{R}^k_+$, $T_n : \mathbb{R}^k_+ \to \mathbb{R}^k_+$, whenever for arbitrary nonzero starting points $x_1, y_1 \in \mathbb{R}^k_+$ and $x_n, y_n$ defined by $x_n = T_{n-1} \cdots T_2 \cdot T_1(x_1)/p(T_{n-1} \cdots T_2 T_1(x_1))$, $y_n$ analogously with $y_1$ instead of $x_1$, the sequence $x_n - y_n$ tends to $0$ for $n \to \infty$ in the Euclidean topology.

**Theorem 4 (Weak ergodicity for nonlinear operators).** Let $(T_n)_n$ be a sequence of operators on $\mathbb{R}^k_+$ that are proper, subhomogeneous, and ray-preserving. Suppose there is some $r \geq 1$ such that the sequence of lumped operators $(S_m)_m$ defined by $S_m = T_{m+1} \cdots T_{m+1}$, $T_m$ is uniformly ascending and uniformly pointwise bounded. Then there holds weak ergodicity for the sequence $(T_n)_n$.

**Proof.** The theorem will be a consequence of Theorem 1. Let $X = \{x \in \mathbb{R}^k_+ \mid p(x) = 1\}$ and put $f_n = T_n$. By Lemma 1(ii) $S_m = f_{m+1} \cdots f_{m+1} \cdots f_{m+1}$, $f_m = F_m$, $F_m$ being a lumped operator for $(f_n)_n$. Since $(S_m)_m$ is assumed to be uniformly bounded, each $S_m$ has to be subhomogeneous. Hence by the assumption of uniform pointwise boundedness from Lemma 1(iv) the existence of $a, b \in \mathbb{R}^k_+ \mid a > 0$ follows such that $F_m(X) \subseteq Y$ for $Y = \{x \in X \mid a \leq x \leq b\}$. Because of $a > 0$, the first inequality of Lemma 2 yields $s = \sup \{\mu(x, y) \mid x, y \in Y\} < +\infty$. Truncating $\mu$ as $d(x, y) = \min(\mu(x, y), s)$ for $x, y \in X$, we obtain the metric space $(X, d)$. Since $Y$ is compact in the Euclidean topology ($p$ was assumed to be continuous) according to Lemma 2, $Y$ is compact also in $(X, d)$, $f_n$ is nonexpansive on $(X, d)$ because of Lemma 1(i). Furthermore, by Lemma 1(iii) $\mu(F_m(x), F_m(y)) \leq c(x, y)$ for all $x, y \in X$ and $c(x, y) < d(x, y)$ for all $x \neq y$ with $\mu(x, y) < +\infty$ where $c(x, y) = -\log \{\varphi(\lambda(x, y)) \cdot \varphi(\lambda(x, y))\}$. Thus we obtain for all $x, y \in Y$ $d(F_m(x), F_m(y)) \leq c(x, y)$ and $c(x, y) < d(x, y)$ provided $x \neq y$. That is, $(F_m)_m$ is contractive on $Y$ and we may apply Theorem 1. By this theorem together with Lemma 2 $x_n - y_n \to 0$ in the Euclidean topology, whereby

$$x_n = f_{n-1} \cdots f_2 \cdot f_1(x_1) = \frac{T_{n-1} \cdots T_2 \cdot T_1(x_1)}{p(T_{n-1} \cdots T_2 \cdot T_1(x_1))}$$

and $y_n$ analogously. The starting points $x_1, y_1$ are arbitrary in $X$, but since the $T_i$ are ray-preserving we may allow for arbitrary nonzero starting points in all of $\mathbb{R}^k_+$. □

An interesting special case of the theorem is obtained if concave operators are considered. An operator $T : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ is concave whenever $T(\lambda x + (1 - \lambda) y) \geq \lambda T x + (1 - \lambda) T y$ for any $x, y \in \mathbb{R}^k_+$ and any $\lambda \in [0, 1]$.

**Corollary (Concave version of the Coale-Lopez theorem).** Consider a scale induced on $\mathbb{R}^k_+$ by a vector space norm. There holds weak ergodicity for every sequence $(T_n)_n$ of proper, ray-preserving and concave operators on $\mathbb{R}^k_+$, provided some sequence of lumped operators $S_m = T_{m+1} \cdots T_{m+1}$, $T_m$ is uniformly pointwise bounded.

**Proof.** To obtain the corollary from Theorem 4 we show that $T_n$ is subhomogeneous and that $(S_m)_m$ is ascending. Consider a concave operator $T$ on $\mathbb{R}^k_+$ and let $\lambda x \leq y$ for $x, y \in X$ and $0 \leq \lambda < 1$. For $z = y - \lambda x$, $z \succeq 0$ and $y = \lambda x + (1 - \lambda)(z/(1 - \lambda))$. Concavity implies $Tz \leq \lambda T x + (1 - \lambda)((z/(1 - \lambda))$ in particular $Tz \leq \lambda T x$ which by approximation is seen to be true also for $\lambda = 1$. Hence each $T_i$ is subhomogeneous. Suppose now for all $x \in X$ and some $a, b \in \mathbb{R}^k_+$, $a > 0$, $b \leq T x \leq b$. By subhomogeneity of $T$

$$T\left(\frac{z}{1 - \lambda}\right) = T\left(\frac{p(z)}{1 - \lambda}\cdot\frac{z}{p(z)}\right) \leq \frac{p(z)}{1 - \lambda} \cdot T\left(\frac{z}{p(z)}\right)$$

and therefore $Ty \leq \lambda Tx + p(z) T(z/p(z))$. Since $p$ is induced by a norm, $p(y) = p(\lambda x + z) \leq \lambda p(x) + p(z)$ and hence $p(z) \geq 1 - \lambda$. Furthermore, there is a real number $0 < s \leq 1$ such that $su \leq y$ and therefore $sTx \leq s \leq u \leq T(z/p(z))$. Thus we obtain $Ty \leq \lambda Tx + (1 - \lambda) sTx = \varphi(\lambda) T x$, with $\varphi(\lambda) = \lambda + (1 - \lambda)$. This formula is true by approximation also for $\lambda = 1$. Being a composite of concave operators $S_m$ is concave.
too and by the uniform pointwise boundedness there exist $u, v \in \mathbb{R}_+^k$, $u > 0$, such that $u \leq S_m x \leq v$ for all $x \in X$ and all $m$. Putting $T = S_m$ we conclude that $\lambda x \leq y$ for $x, y \in X$ and $\lambda \in [0, 1]$ implies $\varphi(\lambda) S_m x \leq S_m (y)$ for all $m$, whereby $\varphi(\lambda) = \lambda + (1 - \lambda) u$. This shows that $(S_m)_m$ is ascending. □

The weak ergodicity theorem or Coale-Lopez theorem referred to in the literature (cf. [4], [8], [13], [14]) is contained in the above corollary as the special case of linear operators $T_n$. In that special case weak ergodicity holds provided the $T_n$ are proper and $(S_m)_m$ is uniformly pointwise bounded. For example, this is guaranteed if all $S_m$ are strictly positive (for some $r \geq 1$) and all the possible entries (all the possible nonzero entries) of the matrices $T_n$ for $n = 1, 2, \ldots$, are bounded from above (bounded from below by some positive constant) (cf. [8], [14]; in [8] weak ergodicity is not with respect to the Euclidean topology but with respect to the topology belonging to Hilbert's metric).

The concept of strong ergodicity also stems from the theory of Markov chains. Generalizing, we say there holds strong ergodicity (relative to some fixed scale $p$) for a sequence $(T_n)_n$ of operators on $\mathbb{R}_+^k$, when for arbitrary nonzero starting points $x_1 \in \mathbb{R}_+^k$ the sequence defined by

$$x_n = \frac{T_{n-1} \cdots T_2 \cdot T_1(x_1)}{p(T_{n-1} \cdots T_2 \cdot T_1(x_1))}$$

converges in the Euclidean topology to the same limit point $x^*$.

**Theorem 5 (Strong ergodicity for nonlinear operators).** Let $(T_n)_n$ and $(S_m)_m$ be sequences of operators on $\mathbb{R}_+^k$ as in Theorem 4. Suppose in addition for the lumped operators $S_m$ uniform convergence on $X = \{x \in \mathbb{R}_+^k | p(x) = 1\}$ equipped with the Euclidean metric to some operator $S$ on $\mathbb{R}_+^k$. Then there holds strong ergodicity for the sequence $(T_n)_n$ and the limit point $x^*$ is the unique eigenvector of $S$ in $X$.

**Proof.** Putting $f_n = T_n, F_m = S_m$ the assumptions of Theorem 1 are satisfied according to the proof of Theorem 4. To apply Theorem 2 we show uniform convergence of $F_m$ to $F = S$ on the metric space $(X, d)$ with $d(x, y) = \min (\mu(x, y), s)$ being the truncated Hilbert metric (as in the proof of Theorem 4). Since by assumption $(S_m)_m$ is uniformly pointwise bounded it follows that $S_m x \leq u > 0$ for all $m$, all $x \in X$ (as in the proof of part (iv) of Lemma 1). In particular $S x \leq u > 0$ for all $x \in X$ and $F = S$ is well defined on $X$. Because of Lemma 2 (and the remark thereafter) and because of $\mu(S_m x, S x) = \mu(F_m(x), F(x))$ for $x \in X$ the uniform convergence of $S_m$ to $S$ for the Euclidean metric implies uniform convergence of $F_m$ to $F$ for the metric $d$. Theorem 2 yields convergence with respect to $d$ of

$$x_n = f_{n-1} \cdots f_2 \cdot f_1(x_1) = \frac{T_{n-1} \cdots T_2 \cdot T_1(x_1)}{p(T_{n-1} \cdots T_2 \cdot T_1(x_1))}, \quad x_1 \in X,$$

to the unique fixed point $x^*$ of $F$ in $X$. By Lemma 2 convergence is also with respect to Euclidean topology. Obviously fixed points of $F$ in $X$ correspond in a unique manner to eigenvectors of $S$ in $X$. Because the $T_n$ are ray-preserving, $x_1 \in X$ may be replaced by any nonzero $x_1 \in \mathbb{R}_+^k$. This proves strong ergodicity of $(T_n)_n$. □

As for weak ergodicity we may specialize to concave operators.

**Corollary (Strong ergodicity for concave operators).** Consider a scale induced on $\mathbb{R}_+^k$ by a vector space norm. There holds strong ergodicity for every sequence $(T_n)_n$ of proper, ray-preserving, and concave operators on $\mathbb{R}_+^k$ provided some sequence of lumped operators $(S_m)_m$ is uniformly pointwise bounded and converges uniformly on $X = \{x \in \mathbb{R}_+^k | p(x) = 1\}$ equipped with the Euclidean metric to some operator $S$ on $\mathbb{R}_+^k$. 849
Proof. \((T_n)_n\), \((S_n)_m\) satisfy the assumptions of Theorem 4, according to the proof of the corollary following Theorem 4. Hence the above corollary is implied by Theorem 5. \(\Box\)

Using Theorem 3 of the previous section, we may obtain sufficient conditions for strong ergodicity without referring to lumped operators. For this from matrix theory we borrow the following notion. An operator \(T: \mathbb{R}^+ \to \mathbb{R}^+_t\) is primitive (for \(p\)), if there exists \(r \geq 1\) such that for any \(x, y \in X = \{x \in \mathbb{R}^+_t | p(x) = 1\}\) and \(\lambda \in \mathbb{R}_+\), \(\lambda x \leq y\), \(\lambda x \neq y\) implies that \(\lambda T^r x < T^r y\).

**Theorem 6.** Let \(T_n: \mathbb{R}^n_+ \to \mathbb{R}^n_+, n = 1, 2, \ldots, \) be proper, subhomogeneous, and ray-preserving operators that converge, uniformly on the intersection with the unit sphere of some monotonic norm \(\| \cdot \|\), to an operator \(T\) on \(\mathbb{R}^+_t\). Suppose the operator \(T\) is proper, ray-preserving, continuous, and primitive (for \(\| \cdot \|\)). Then there holds strong ergodicity for \((T_n)_n\) with scale \(\| \cdot \|\) and the limit point \(x^*\) is the unique normalized eigenvector of \(T\).

Proof: Let \(X = \{x \in \mathbb{R}^n_+ | \|x\| = 1\}\) equipped with \(e(x, y) = \|x - y\|\) is a compact metric space on which \((T_n)_n\) converges uniformly to an operator \(T\) which must be uniformly continuous. Formula (2) of step (3) in the proof of Theorem 2 then yields that on \((X, e)\) the sequence of lumped operators \(S_n = T_{m \rightarrow 1} \cdots T_{m+1} T_{m+1} \cdots T_{m+1}\) converges uniformly to \(T^r\) (\(r\) as in the definition of primitivity). Take \(\| \cdot \|\) as scale and put \(f_n = T_n\). From Lemma 1 for the lumped operators \(F_n\) to \((f_n)_n\) \(F_n = S_n\) and \(F = T\). From Lemma 2 (and the remark thereafter) it follows that on \(X\) \(F_n\) converges uniformly to \(F\) with respect to Hilbert's metric \(\mu\). We shall apply Theorem 3 for \(U = \{y \in X | \|a < y < 2b\}\). Obviously, \(U\) is relatively open in \(X\) for the Euclidean topology and the closure \(\bar{U} = \{y \in X | \|a \leq y \leq 2b\}\) is compact. For \(d(x, y) = \min (\mu(x, y), s), s = \sup \{\mu(x, y) | x, y \in U\}\) from Lemma 2 it follows that \(U\) is open in \((X, d)\), the closure of \(U\) in this space equals \(\bar{U}\) and \(U\) is compact in \((X, d)\). Furthermore \(F(X) \subseteq U\). Finally, from the primitivity of \(T\) it follows (as for part (iii) of Lemma 1) that \(\mu(F(x), F(y)) < \mu(x, y)\) for \(x, y \in U\). Hence \(F\) is contractive on \(\bar{U}\) for \(d\).

Thus Theorem 3 yields the convergence relative to \(d\) of the sequence defined by \(x_n+1 = f_n(x_n), x_1 \in X\), to some \(x^* \in X\). Also, \(x^*\) is the unique fixed point of \(F\) in \(X\). Primitivity of \(T\) implies \(x^* = T^r x^*/\|T^r x^*\| > 0\). By Lemma 2 \((x_n)_n\) converges to \(x^*\) in the Euclidean topology. Since the \(T_n\) are ray-preserving, any nonzero \(x_1 \in \mathbb{R}^n_+\) is allowed to be a starting point. Thus \((T_n)_n\) is strongly ergodic. Concerning the assertion on \(x^*\) in the theorem, it suffices to show \(x^*\) to be an eigenvector of \(T\). Since \(x^* > 0\), there exists \(0 < \lambda\) sufficiently small with \(\lambda \|T^{-1} x^*\|/\|T^{-1} x^*\| \leq x^*\). Being the limit of subhomogeneous operators, \(T\) is subhomogeneous and therefore \(\lambda T(T^{-1} x^*)/\|T^{-1} x^*\| \leq T x^*\). Since \(T\) is ray-preserving it follows that \(\mu(T x^*) \leq T x^*\) for some \(\mu > 0\) and hence \(T x^* > 0\). Because of \(T_n x_n \rightarrow T x^*\) in the Euclidean topology by Lemma 2 it follows that \(f_n(x_n) \rightarrow f(x^*)\) for \(d\). But \(f_n(x_n) = x_{n+1} \rightarrow x^*\), and hence \(f(x^*) = x^*\), i.e., \(x^*\) is an eigenvector of \(T\). \(\Box\)

As a simple special case the theorem on strong ergodicity for nonnegative matrices (cf. [14]) follows directly from Theorem 6. For \(T_n\) linear the only assumptions placed on \(T_n\) by Theorem 6 are properness (e.g., fulfilled if there is no column of zeros) and pointwise convergence of \(T_n\) to an operator \(T\) some power of which is strictly increasing.

4. Example: A density- and time-dependent Leslie model. Consider a population of female individuals at discrete points of time. The growth of the population is
dependent upon birth and death which are, among others, age-specific. Hence the population is divided into a finite number of age groups, say 1, 2, \ldots, k. Denote by $X_i(t)$ the number of individuals in age group $i \in \{1, \ldots, k\}$ at time $t \in \{1, 2, \ldots\}$ and let $X(t) = (X_1(t), \ldots, X_k(t))$ be the population vector at time $t$. The total population at time $t$ is given by $\|X(t)\|$, where $\|\cdot\|$ is the norm on $\mathbb{R}^k$ defined by $\|x\| = \Sigma x_i$. The vector $x(t) = X(t)/\|X(t)\|$ is called the age structure at time $t$. The birth rate in age group $i$ is denoted by $b_i(t, X(t))$ and may depend on time and the population vector. The survival rate in age group $i$, denoted by $s_i(t, X(t))$, specifies the proportion of $X_i(t)$ surviving to be in age group $i+1$ at time $t+1$. The survival rate $s_k(t, X(t))$ for the oldest group therefore is 0. It is assumed that all the other survival rates and the birth rates are strictly positive. Thus we arrive at the following model:

$$X_i(t+1) = \sum_{i=1}^k b_i(t, X(t))X_i(t),$$

$$X_{i+1}(t+1) = s_i(t, X(t))X_i(t) \quad \text{for } i = 1, \ldots, k-1.$$  

Or, equivalently,

$$X(t+1) = T(t)X(t) \quad \text{where } T(t) : \mathbb{R}^k \rightarrow \mathbb{R}^k$$

is given by

$$T(t)x = L(t, x)x = \begin{bmatrix} b_1(t, x) & \cdots & b_k(t, x) \\ s_1(t, x) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_{k-1}(t, x) & 0 \\ 0 & \cdots & 0 & s_k(t, x) \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \\ x_k \end{bmatrix}.$$

This is a generalized Leslie model where birth rates and survival rates are allowed to depend on time and on density $X(t)$. In the original Leslie model (cf. [4], [8], [13], [14]) birth rates and survival rates are assumed to be constant, and hence the Leslie matrix $L(t, x)$ is constant, $L(t, x) = L$ for all $t$ and $x$. Since $L$ turns out to be a primitive matrix this case is already covered by a theorem on matrices due to Perron which implies that the age structure $x(t)$ converges to an equilibrium $x^*$ (cf. [12], [14]).

The case that $L(t, x)$ is time-dependent only, $L(t, x) = L(t)$, is covered by the (linear) Coale-Lopez theorem which yields weak ergodicity of the age structure $x(t)$ (cf. [4], [8], [13], [14]). However, if $L(t, x)$ also depends on $x$ and therefore $T(t)$ becomes a nonlinear operator, a lot of things may happen. So it is shown in [9] for the very simple model given by $k = 2$, $b_i(t, x) = b_i \exp [-a(x_1 + x_2)]$, $s_i(t, x) = \text{constant}$, that chaotic dynamics occurs for certain choices of the parameters $b_1$, $b_2$, $a$.

Being interested in weak ergodicity also for the nonlinear case we therefore have to make some assumptions which constitute our concave Leslie model.

The concave Leslie model is based on the following assumptions.

1. The functions $x \rightarrow b_i(t, x)x_i$, $x \rightarrow s_i(t, x)x_i$ are concave on $\mathbb{R}^+_x$ for all $i, t$.
2. $b_i(t, \lambda x)/b_i(t, x) = s_i(t, \lambda x)/s_i(t, x)$ for all $\lambda > 0$ and all $i, j, x$.
3. There exist functions $c(\cdot)$, $d(\cdot) : \mathbb{R}^+_x \rightarrow \mathbb{R}^+_x \setminus \{0\}$ with $c(\cdot)$ not increasing, $d(\cdot)$ not decreasing with respect to "$\leq$" such that

$$c(x) \leq b_i(t, x), s_i(t, x) \quad \text{and} \quad b_i(t, x)x_i, s_i(t, x)x_i \leq d(x)$$

for all $i$, $t$ and $x$.

We show that the concave Leslie model satisfies the assumptions of the concave version of the Coale-Lopez theorem presented in the last section. Assumption (1)
obviously implies that all the operators $T(t) = L(t, x)x$ are concave. The assumption itself means that the number of births or survivals contributed by a particular age group decreases by "population pressure" with the density in that group. Assumption (2) implies $L(t, \lambda x) = \mu L(t, x)$ for some $\mu$, which may depend on $\lambda$, $t$, $x$ and hence every $T(t)$ is ray-preserving. By this assumption a certain "homogeneity" of the vitality rates with respect to changes in total population is required. The assumption is met, e.g., if all vitality rates are homogeneous of the same degree with respect to total population. The assumption (3) requiring uniform upper and lower bounds for the vitality rates implies obviously that the $T(t)$ are proper and yields the existence of a uniformly pointwise bounded sequence of inhomogeneous iterates as shown by the following lemma.

**Lemma 3.** By assumption (3) there exist for every $x \in \mathbb{R}^k_+ \setminus \{0\}$ $u(x), v(x) \in \mathbb{R}^k_+$ where $u(x) > 0$ such that

$$u(x) \equiv T(m + k - 1) \cdots T(m + 1) \cdot T(m)x \leq v(x) \quad \text{for all } m \in \mathbb{N}.$$ 

**Proof.** By (3) $T(t)x \leq w(x)$ for all $t$, where $w(x) = (kd(x), d(x), \cdots, d(x))$ and $w(\cdot)$ is not decreasing with respect to "\leq." Hence $T(t)T(s)x \leq w(T(s)x) \leq w \cdot w(x)$. By iteration, if $S(m) = T(m + k - 1) \cdots T(m + 1) \cdot T(m)x \leq w \cdots w(x)$. Choose $v(x) = w^k(x) = w \cdots w(x)$. Furthermore by (3) $L(t, x) \geq c(x) L$, where

$$L = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Hence $T(t)x = L(t, x)x \geq c(x) L x$ for all $t$. It follows, by using $T(s)x \leq w(x)$, that

$$T(t)T(s)x \leq c(T(s)x)c(x)Lx \equiv c(T(s)x)c(x)L^2x \equiv (w(x))c(x)L^2x.$$ 

Therefore by iteration for $S(k)$ defined above

$$S(m)x \geq c(w^{k-1}(x)) \cdots c(w(x))c(x)L^k x.$$ 

Choose as $u(x)$ the right-hand side of this inequality. It is easily checked that the matrix $L^k$ is strictly positive. Hence $u(x) > 0$. □

Due to the lemma we can apply the concave version of the Coale–Lopez theorem (corollary to Theorem 4) which yields that weak ergodicity holds in the above concave Leslie model. A simple example of a concave Leslie model is the following one that contains the linear time-dependent Leslie model as a special case. Let for $k$ arbitrary and $i \in \{1, \cdots, k\}$, $t \in \{1, 2, \cdots\}$,

$$b_i(t, x) = b_i(t)x_i^{-1}, \quad s_i(t, x) = s_i(t)x_i^{-1},$$

where $0 < \alpha \leq 1$ and $c_i \leq b_i(t), s_i(t) \leq c_i$ for all $i$, all $t$ with certain positive constants $c_i$. Assumptions (1) and (2) for a concave Leslie model are obviously satisfied. Assumption (3) is fulfilled by choosing the following functions:

$$c(x) = c_1 \min_i x_i^{-1}, \quad d(x) = c_2 \sum_i x_i \quad \text{for } x \in \mathbb{R}^k_+ \setminus \{0\}.$$ 

The linear Leslie model is obviously contained for $\alpha = 1$. For $\alpha < 1$, e.g., $\alpha = \frac{1}{2}$, the operator $T(t)$ is neither positive homogeneous nor additive. Nevertheless, as for the linear case, weak ergodicity holds for this simple nonlinear example.
The example may serve also to illustrate the general Theorem 6 on strong ergodicity. For this suppose that for all \(i, b_i(t)\) and \(s_i(t)\) tend to \(b_i > 0\) and \(s_i > 0\), respectively, as \(t \to \infty\). If

\[
L(t) = \begin{bmatrix}
  b_1(t) & \cdots & b_k(t) \\
  s_1(t) & 0 & \cdots \\
  0 & \ddots & \ddots \\
  0 & \cdots & s_{k-1}(t) & 0
\end{bmatrix}, \quad L = \begin{bmatrix}
  b_1 & \cdots & b_k \\
  s_1 & 0 & \cdots \\
  0 & \ddots & \ddots \\
  0 & \cdots & s_{k-1} & 0
\end{bmatrix},
\]

then \(T(t)x = L(t)x^a\) and \(T(t)\) converges uniformly on \(X = \{x \in \mathbb{R}_+^k : \|x\| = 1\}\) to the operator \(T\) given by \(Tx = Lx^a\). Obviously, \(T(t)\) and \(T\) are proper and \(T\) is continuous on \(X\). \(T(t)\) is also ray-preserving and subhomogeneous because of

\[
T(t)(\lambda x) = L(t)(\lambda x)^a = \lambda^a L(t) x^a = \lambda^a T(t)x.
\]

The same applies to \(T\). To see primitivity let \(\lambda x \leq y, \lambda x \neq y\) for \(x, y \in X\). Successive application of \(L\) yields, using the monotonicity of \(x \to x^a\), that \(T^k(\lambda x) < T^k y\). Hence \(\lambda T^k x \leq \lambda^a T^k x = T^k(\lambda x) < T^k y\), and \(T\) is primitive. Thus Theorem 6 supplies strong ergodicity for this little nonlinear example, a result that still contains the corresponding result for the linear case.

REFERENCES