Brouwer Fixed Point Theorem: A Proof for Economics Students

Takao Fujimoto

Abstract This note is one of the efforts to present an easy and simple proof of Brouwer fixed point theorem, which economics students can, hopefully, grasp both in terms of geometry and through its economic interpretation. In our proof, we use the implicit function theorem and Sard’s theorem. The latter is needed to utilize a global property.

1 Introduction

In this note, we give a proof of Brouwer fixed point theorem ([2]), based on the method in Kellogg, Li and Yorke ([7]). Our idea consists in using a special set for a given map and a special boundary point to start with. Two theorems are used in our proof, i.e., the implicit function theorem and Sard’s theorem ([12]). (See Golubitsky and Guillemin ([4]) for a result as we use in this note.) Unfortunately, we still need Sard’s theorem, which seems to be somewhat beyond the mathematical knowledge of average economics students.

In section 2, we describe our proof, and in the following section 3, an economic interpretation of the proof is given. The final section 4 contains several remarks.

*Faculty of Economics, Fukuoka University
2 Proof

Let $f(x)$ be a map from a compact convex set $X$ into itself. We assume that $f(x)$ is twice continuously differentiable and that $X$ is a set in the $n$-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 1$) defined by

$$X \equiv \{x | x \geq 0, \sum_{i=1}^{n} x_i \leq 1, x \in \mathbb{R}^n\}.$$ 

Then define a direct product set $Y$

$$Y \equiv X \times T, \text{ where } T \equiv [0, \infty).$$

The symbol $J_f(x)$ denotes the Jacobian matrix of the map $f(x)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Fig. 1}
\end{figure}

Now we begin our proof. First of all, as the hypothesis of mathematical induction, we suppose that the theorem is true when the dimension is less than $n$. (When $n = 1$, it is easy enough to show the existence of at least one fixed point.) Let us consider the set

$$C \equiv \{y | f(x) - t \cdot x = 0, y = (x, t) \in Y\}.$$
(See Fig.1 above and Fig.2 below.) By the implicit function theorem, when the determinant

$$|J_f(x) - t \cdot I| \neq 0$$

at a point \((x,t)\), there exists a unique curve passing through the point in one of its neighborhoods. If the origin 0 is a fixed point, the proof ends, and so, let us suppose not. Then, there is a point \(y^0 \equiv (x^0, t^0)\) in a neighborhood of the origin such that \(y^0 \in C\), because of the inductive hypothesis. On the other hand, in the set \(B \equiv \{x \mid x \geq 0, \sum_{i=1}^{n} x_i = 1, x \in \mathbb{R}^n\}\), there is either a fixed point or a point in \(C\) with \(t < 1\). (Thus, when the Jacobian \(|J_f(x)|\) has no eigenvalue either in the interval \([0,1)\) or \((1,t^0]\), we can find out a curve in \(C\) which includes a point with \(t = 1\), i.e., a fixed point.)

![Fig.2](image)

When there is no fixed point, we can construct a continuously differentiable map \(g(x)\) from \(X\) to its boundary \(\partial X\) as the point where a line \(x - f(x)\) hits the boundary on the side of \(x\). The Jacobian \(|J_g(x)|\) surely vanishes, and yet its rank is \((n-1)\) almost
everywhere because of Sard’s theorem. (Note that now \( n \geq 2 \).) We suppose without the loss of generality that the origin is one of the points where the rank of \( |J_g(x)| \) is indeed \( (n-1) \). This means that any curve contained in \( C \) is locally unique as is shown in Kellogg, Li and Yorke ([7]).

So, starting from the initial point \( y^0 \), we extend the curve which satisfies \( f(x) - t \cdot x = 0 \). This can be done by considering a solution of differential equation with respect to arc length \( s \).

\[
\begin{cases}
\frac{dx}{ds} = (J_f(x) - t \cdot I)^{-1} \cdot x \cdot \frac{dt}{ds} & \text{when } |J_f(x) - t \cdot I| \neq 0, \\
(J_f(x) - t \cdot I) \cdot \frac{dx}{ds} = 0 & \text{and } dt = 0 \text{ when } |J_f(x) - t \cdot I| = 0.
\end{cases}
\]

The above equation can be derived by differentiating

\( f(x) - t \cdot x = 0 \)

with respect to arc length \( s \) along the curve, that is,

\[
J_f(x) \frac{dx}{ds} - \frac{dt}{ds} x - t \frac{dx}{ds} = 0.
\]

If there is no fixed point, this curve remains within a subset of \( C \) whose \( t \in [a, b] \), where \( 1 < a \) and \( b < t^0 \), having an infinite length without crossing. (Note that there is a minor flaw in the proof of Theorem 2.2 in Kellogg, Li and Yorke ([7, p.477]).) This leads to a contradiction because there is at least one accumulation point where the local uniqueness mentioned above is lost. Hence there should be a fixed point.

When \( f(x) \) is merely continuous, we can prove the theorem as is done in Howard ([5]), which expounds Milnor ([9]) and Rogers ([11]).

### 3 Economic Interpretation

We can interpret our method of proof in a simple share game among \( (n + 1) \) players. Let us suppose that in our economy there exists
one unit of a certain commodity which every individual player, numbered from 0 to \( n \), wishes to have. When a point \( x \) of \( X \) is given, it represents the shares \( p_i \)'s of players: \( p_0 = 1 - \sum_{i=1}^{n} x_i \) and \( p_i = x_i \) for \( i = 1, \ldots, n \). Staying at the origin implies the whole commodity belongs to player 0. A given map \( f(x) \) stands for a rule of changes in players’ shares. For a point \( x \) to be in \( C \), it is required that after changes in shares by \( f(x) \), the ratios of shares among the players from 1 to \( n \) should remain the same as before, while the share of player 0 may increase or decrease. (See Fig. 1 above.)

We know that in a neighborhood of the origin there is a point in \( C \), provided that the origin is not a fixed point. Starting from this point, we grope for successive neighboring points in \( C \). As we continue the search, we reach a fixed point, a special point where no changes are made after the transformation by \( f(x) \). This is because, otherwise, we would get into a path of an infinite length with no crossing point within a compact set, which yields a contradiction to the local uniqueness guaranteed by use of Sard’s theorem.

4 Remarks

It seems that the theorem was first proved in 1904 by Bohl ([1]) for the case of dimension 3, and in the general case by Hadamard before or in 1910, whose proof was published in an appendix of the book by Tannery ([15]). It is reported that Hadamard was told about the theorem by Brouwer (Stuckless ([14])). Thus, it may be more appropriate to call the theorem \( B^2H \) theorem as astronomers do. (Stuckless ([14]) also mentions the contributions by Bolzano (1817) and Poincaré (1883) which are equivalent to Brouwer fixed point theorem. Then, more precisely, \( B^3HP \) theorem.)

Differentiableness is useful to tame down possibly wild movements of continuous functions, and yet it is, in normal settings, powerful only to derive local properties. In order to prove Brouwer fixed point theorem, we need a gadget related to global properties:
(1) simplicial subdivisions and Sperner’s lemma ([13]) and KKM theorem ([8]),
(2) Sard’s theorem, or
(3) volume integration as used by Kannai ([6]) and Howard ([5]).

As in Kellogg, Li and Yorke ([7]), we can devise out a numerical procedure to compute a fixed point. First note that

\[(J_f(x) - t \cdot I) \cdot dx = x \cdot dt\]

by the total differentiation of the equation \(f(x) - t \cdot x = 0\). Thus, starting from the point \(y^0\), we proceed \(\Delta x = (J_f(x) - t \cdot I)^{-1} \cdot hx\) when \(|J_f(x) - t \cdot I| \neq 0\), and \(\Delta x = hv\), where \(v\) is a real eigenvector of \(J_f(x)\) with its eigenvalue being \(t\), when \(|J_f(x) - t \cdot I| = 0\). Here, \(h\) is a small positive scalar of step-size. It is clear that near a fixed point, where \(t = 1\), this numerical procedure is quasi-Newton-Raphson method.

Originally, the author tried to obtain a proof by extending the notion of fixed point, and by using the result in Fujimoto ([3]). This lead to the method in Kellogg, Li and Yorke ([7]). When we define two sets as

\[
CG \equiv \{ y \mid f(x) - t \cdot x \geq 0, \ t \geq 1, \ y = (x, t) \in Y \}, \quad \text{and}\\
CL \equiv \{ y \mid f(x) - t \cdot x \leq 0, \ t \leq 1, \ y = (x, t) \in Y \},
\]

these sets certainly have respective points in \(Y\) with \(t = 1\). If the two sets \(CG\) and \(CL\) intersect with each other, there exists a fixed point.

The reader is referred to Park and Jeong ([10]) for an interesting circular tour around Brouwer fixed point theorem.

In this note, we have shown a proof of Brouwer fixed point theorem, using the implicit function theorem and Sard’s theorem, and thus twice continuous differentiability of the map. To be understandable to economics students, it is desirable to establish the theorem without depending upon Sard’s theorem. I wish to have such proofs and present them in the near future.
References


