Non-Substitution with Joint Production

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Abstract

We show that empirical input-output models with joint production exhibit a very simple mathematical structure, with similar features to the standard single-production ones. Namely, the existence of non-negative solutions for all admissible parameter values, and the non-substitution property.

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1 Introduction

The input-output model is a conventional tool for the analysis of a multisectedored economy which has been shown most fruitful. It is based on the treatment of the production side of the economy as a finite collection of economic sectors or “industries”, each of which is the exclusive producer of a specific “commodity” (single production). This modeling strategy requires the aggregation of commodities and firms into a relatively small number so that we can identify the industry with the commodity produced. Hence we find sectors that produce “cars”, “telephone services”, or “insurance”, to name just three.

Assuming that the economy exhibits constant returns to scale (linear transformations), one can estimate empirically the technical coefficients of production with the data corresponding to the actual production and input use. The model enables us to analyze the quantitative impact of changes in final demands and added values on industrial production levels and relative prices. There is a key theoretical result that ensures the validity of this approach: the Non-Substitution property [Samuelson (1951), Georgescu-Roegen (1951)]. This property establishes that a single technique can be taken as a sufficient representation of the technology, in spite of the availability of many alternative production processes. This is so because all efficient production plans can be obtained by means of a single technique and because equilibrium prices are uniquely determined. Therefore, computing the data of the technique actually in use is enough to capture all relevant technological information [see for instance Arrow & Hahn (1971, ch. 2), Cornwall (1984, ch.2)].

Joint production is a widespread phenomenon which appears under a variety of forms and is hard to disregard. Most of the industries actually produce a collection of variants of the same type of product (think of the car industry, for instance), or genuinely different goods (e.g., firms that produce home appliances). Note that the demands of those different goods produced by the same industry may behave quite differently and would therefore call for a joint production model. Moreover, the presence of by-products is a general fact and, in particular, it includes the case of used capital which can only be properly modelled as a joint product.

Allowing for joint-production, however, introduces some important difficulties in the model. The very notion of “economic sector” becomes problematic as now the same commodity can, in principle, be produced by several industries. It seems therefore theoretically more accurate to model production in terms of activities, defined as particular combinations of input-output proportions, as in the standard general equilibrium model. Yet this modelling choice affects the analytics of the linear production model: The solvability of the associated equilibrium equation systems is far from trivial and we can not invoke the non-substitution property. Schefold (1978), Bidard (1991, Part II), or Peris & Villar (1993) provide some results on the solvability of input-output models with joint production. Johansen
(1972) and Herrero & Villar (1988) present extensions of the Non-Substitution Theorem to this context. Villar (2003) offers a more general model in which these two aspects are analyzed together with the measurement of productivity differences. The conditions that ensure these results are both compelling and restrictive in the theoretical model.

The purpose of this paper is to show that those difficulties practically vanish if one keeps the key feature of the standard input-output model. Namely, modelling the production side of the economy as a finite collection of economic sectors each of which is the exclusive producer of a specific subset of commodities. This seems the most natural modelling strategy in empirical input-output analysis: keeping the notion of “industry” by a suitable aggregation of commodities and firms, and yet allowing for joint production within these industries. Hence we would be able to model the car industry as producing, e.g., utility cars, luxury cars, 4WD cars, or vanettes; the output of the telephone industry as including different products such as domestic and long-distance cable telephone calls, mobile telephone and internet services, etc. Plus, of course, used fixed capital and some other by-products (which may or may not be industry specific).

We say that an economy is sector specific when each sector is the exclusive producer of a subset of commodities (those that identify it). We show that a sector specific economy with joint production in which all commodities can be simultaneously produced exhibits a very simple mathematical structure and good operational properties. In particular, we can ensure the applicability of an extended non-substitution property.

The paper is organized as follows. Section 2 contains the basic model whereas section 3 deals with the non-substitution theorem.

2 The model

Consider an economy with $\ell + 1$ goods, consisting of $\ell$ producible commodities and a single primary factor (labour), and $n$ industries or economic sectors which operate under constant returns to scale, each of which may produce more than one net output (joint production). Economic sectors are differentiated by their output. Let $L = \{1, 2, ..., \ell\}$ denote the index set of producible commodities and $N = \{1, 2, ..., n\}$ that corresponding to industries.

Consider now the following definition:

**Definition 1** A sector specific economy is one in which each sector $j \in N$ is the exclusive producer of a specific subset of commodities $C_j \subset L$.

When the economy is sector specific we can write $L = \bigcup_{j=1}^{n+1} C_j$ and, for all $j = 1, 2, ..., n$, $C_j \neq \emptyset$ with $C_i \cap C_j = \emptyset$, for all $i \neq j$. Each $C_j$ is the set of commodities that identifies industry $j$ as the net producer of those goods. Note that we allow for the existence of goods that can be produced by several industries
simultaneously (those in the set \( C_{n+1} \), on which we impose no restriction). This notion extends that of *principal product* economies, introduced in Peris & Villar (1993).\(^1\) Clearly, a single-production input-output model is an industry-based economy.

Let a sector specific economy and assume initially that each sector has a unique technique available. Under these circumstances allowing for joint production entails naturally the restriction \( n \leq \ell \). In this context, a production plan that is feasible for the \( j \)th sector is a vector \( y^j \in \mathbb{R}^\ell \) of the form:\(^2\)

\[
y^j = [s^j, -q_j]
\]

where \( s^j \in \mathbb{R}^n \) is a net-output vector of producible commodities and \( q_j \geq 0 \) is the amount of labour required. Assume that labour is a necessary input for production, that is, \( q_j > 0 \) whenever \( y^j \) has some positive entry. Let \( y = \sum_{j=1}^n y^j \) denote an arbitrary aggregate efficient production plan, in which \( y^j \neq 0 \) for all \( j \). Then the linearity of the technology permits one to express all aggregate efficient production plans as follows:

\[
y = \sum_{j=1}^n y^j = \left( \begin{array}{c} M \\ -1 \end{array} \right) q
\]

where \( M \) is an \( \ell \times n \) matrix whose \( (i, j) \) entry is given by \( m_{ij} = \frac{s_{ij}}{q_j} \). \( q \) is a column vector in \( \mathbb{R}^n \), and \( 1 \in \mathbb{R}^n \) is the unit (row) \( n \)-vector. That is, the elements of \( M \) are the technical coefficients of production (net production per unit of labour), \( q \) corresponds to a vector of activity levels (measured in terms of labour), and \( 1q \) is the employment level.

An input-output matrix is called productive when all commodities in \( L \) can be simultaneously produced. Formally:

**Definition 2** We say that an input-output matrix \( M \) is productive if there exists some \( x^o \in \mathbb{R}_{++}^n \), such that \( Mx^o >> 0 \).

Let \( d \in \mathbb{R}_{++}^\ell \) be a given final demand vector. To find an equilibrium in the quantity side of this economy consists of finding a solution to the following linear system:

\[
Mx = d, \quad x \in \mathbb{R}_+^n
\]

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\(^1\) A principal product economy is one in which each sector can be identified as the exclusive net producer of a particular commodity, even though it can jointly produce other goods that correspond to used machines or some by-products.

\(^2\) The symbol \( \mathbb{R}^n \) means the real Euclidean space of dimension \( n \) \((n \geq 2)\), \( \mathbb{R}_{++}^n \) the nonnegative orthant of \( \mathbb{R}^n \), and \( \mathbb{R}_{++}^n \) the interior of \( \mathbb{R}_{++}^n \). A given real \( \ell \times n \) matrix is represented by \( M \), whose \( (i, j) \) entry is \( m_{ij} \). The symbol \( e_i \) means the vector \( \mathbb{R}^n \) whose \( i \)-th element is unity with all the other entries being zero. Inequality signs are \( \geq, >, >> \).
Remark 1  For any given \( n \times \ell \) matrix \( M \) the image of \( M \), denoted by \( \text{Im}(M) \), consists of all vectors \( z \in \mathbb{R}^\ell \) such that \( z = My \), for some \( y \in \mathbb{R}^n \) (mind that \( y \) ranges over the whole Euclidean space). It is easy to see that \( \text{Im}(M) \) is a convex cone. Clearly, when \( n \leq \ell \), not all vectors in \( \mathbb{R}^\ell_+ \) are necessarily in the image of \( M \). To see this, take for instance the matrix
\[
M = \begin{bmatrix}
1 & -0.2 \\
1 & -0.3 \\
-0.1 & 1
\end{bmatrix}.
\]

There is no point \( y \) in \( \mathbb{R}^2 \) which allows us to obtain exactly the vector \( z = (1, 1, 1)' \) as \( My \). Therefore the solvability of the quantity system should be considered with respect to those final demand vectors in \( \mathbb{R}^\ell_+ \) which are also in \( \text{Im}(M) \) (the image of \( M \)).

As for the price system, we know that, under constant returns to scale, the only price vectors that are compatible with profit maximization are those that entail zero profits. Therefore, normalizing prices so that the wage rate is equal to one, we can express the equilibrium prices as a solution to the following system:
\[
pM = 1, \quad p \in \mathbb{R}^\ell_+
\]

3 The Non-Substitution Property

Suppose now that for each industry \( j = 1, 2, ..., n \), there exists a collection of alternative production processes \( (m^j(i))_{i=1}^{T_j} \), where \( T_j \) is a finite number and \( m^j(i) \) a column vector with \( \ell \) entries. Now the equilibrium of the economy implies the choice of techniques. Each particular technique corresponds to a specific choice of a production process by each industry. Let \( M \) denote the matrix which includes all available production processes, that is, \( M \) is an \( \ell \times h \) matrix where \( h = \sum_{j=1}^n T_j \). We identify this matrix with the technology of the economy. We say that a production plan \( x \) associated with a particular technique \( M \) is \textit{efficient} if the net production involved, \( Mx \), cannot be obtained using a lesser amount of labour.

The following result is obtained:

\textbf{Theorem 1} Consider a sector specific economy that contains an \( \ell \times n \) input-output matrix \( M \) capable of producing efficiently some final demand vector \( d^0 \in \mathbb{R}^\ell_{++} \). Then:

(i) This technique \( M \) permits the efficient production of all admissible final demand vectors \( d \in \text{Im}(M) \cap \mathbb{R}^\ell_+ \).

(ii) There exists \( p^* > 0 \) such that, \( p^* M = 1^n \) with \( pM \leq 1^h \) (where superscripts \( n, h \) refer to the dimension of the vector).
The following lemma, which is interesting on its own, will facilitate the proof of the Theorem:

**Lemma 1** Let \( M \) be a productive \( n \times \ell \) input-output matrix corresponding to a sector specific economy. Then, for all \( d \in \text{Im}(M) \cap \mathbb{R}_+^\ell \) there is a unique \( x^* \in \mathbb{R}_+^n \) such that \( Mx^* = d \).

**Proof.**

First note that when \( M \) is productive and sector specific, \( x >> 0 \) whenever \( Mx >> 0 \) (namely, producing net output of all commodities requires the concourse of all sectors). This in turn implies that \( M \) is a matrix of rank \( n \). To see that suppose, by way of contradiction, that \( Mx = 0 \) for some \( x \neq 0 \). If \( x > 0 \), construct a vector \( y = x^o - k(x)x \), where \( x^o \) is such that \( Mx^o >> 0 \) (which exists by the assumption of productivity), and \( k(x) = \min_{x_i \neq 0} (x^o_i / x_i) \). Clearly \( My = Mx^o - kMx >> 0 \) where \( y > 0 \) is a vector that contains at least one zero element, contradicting the assumption that all sectors are essential. When \( x \) includes a negative element, construct a vector \( y = x^o + k(x)x \), where \( k(x) = \min_{x_i < 0} (x^o_i / (-x_i)) \). We find also in this case the same kind of contradiction. Therefore, \( Mx = 0 \) implies \( x = 0 \) and \( rk(M) = n \).

Now let us write

\[
M = \begin{bmatrix} C \\ K \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
\]

where \( C \) is an \( n \times n \) matrix of full rank, and \( d_1 \) has \( n \) components (we can always do this after a suitable permutation of rows). The system \( Cx = d_1 \) admits a unique solution \( x^* = C^{-1}d_1 \). Since \( d \in \text{Im}(M) \) we conclude \( Mx^* = d \).

If \( d >> 0 \) the assumptions of the model imply that \( x^* >> 0 \). Suppose that \( d \) contains some zero. Then, let \( \{d^\nu\} \to d \) denote a sequence in \( \text{Im}(M) \) with \( d^\nu >> 0 \) for all \( \nu \) (the productivity assumption and the fact that \( \text{Im}(M) \) is a cone ensures that we can always construct such a sequence). Let \( x^\nu \) denote the solution of the equation \( Mx = d^\nu \). Clearly \( x^\nu >> 0 \) and \( \{x^\nu\} \to x^* \). The continuity of the linear mapping ensures that \( x^* > 0 \).

Let us now prove the Theorem:

Consider the following program:

\[
\begin{align*}
\min & \quad 1^h y \\
\text{s.t.} & \quad My \geq d^o \\
& \quad y \in \mathbb{R}_+^h
\end{align*}
\]

This problem has a solution since the objective function is continuous and the feasible set is non-empty and bounded from below. Moreover, since \( d^o \) can be

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\(^3\)The strategy of the proof follows the former contribution by Chander (1974).
efficiently produced by the technique $\mathbf{M}$, by assumption, Lemma 1 ensures that the solution to this program will take the form $\mathbf{y}^* = (\mathbf{x}^o, 0)$ where $\mathbf{x}^o \in \mathbb{R}_{++}^n$ is such that $\mathbf{d}^o = \mathbf{Mx}^o$ and $\mathbf{0} \in \mathbb{R}^{h-n}$.

Now consider the dual program:
\[
\begin{align*}
\max_{\mathbf{p}} & \quad \mathbf{pd}^o \\
\text{s.t.} & \quad \mathbf{pM} \leq 1^h \\
& \quad \mathbf{p} \in \mathbb{R}^\ell_+
\end{align*}
\]

This program has an optimal solution $\mathbf{p}^*$ and we should have $\mathbf{p}^* \mathbf{M} = 1^n$, i.e. the constraints are strictly binding for the selected process because $\mathbf{x}^o \gg \mathbf{0}$. This proves part (ii) of the Theorem.

Let now $\mathbf{d}$ stand for an arbitrary point in $\text{Im}(\mathbf{M}) \cap \mathbb{R}^\ell_+$. Lemma 1 ensures the existence of a unique $\mathbf{x}^*$ that satisfies $\mathbf{Mx}^* = \mathbf{d}$. Let $\mathbf{G}$ denote an alternative technique which is also able to produce $\mathbf{d}$. That is, there exists $\mathbf{x} \in \mathbb{R}^n_+$ such that $\mathbf{Gx} \geq \mathbf{d}$. We obtain by premultiplying $\mathbf{p}^*$,
\[
\mathbf{p}^* \mathbf{G} \mathbf{x} \geq \mathbf{p}^* \mathbf{d}
\]
Moreover, since $\mathbf{p}^* \mathbf{G} \leq 1^n$, it follows:
\[
\mathbf{p}^* \mathbf{G} \mathbf{x} \leq 1^n x
\]
Then, we have,
\[
1^n x^* = \mathbf{p}^* \mathbf{M} x^* = \mathbf{p}^* \mathbf{d} \leq \mathbf{p}^* \mathbf{G} \mathbf{x} \leq 1^n x
\]
That is to say, the technique $\mathbf{M}$ can efficiently produce all final demand vectors in $\text{Im}(\mathbf{M}) \cap \mathbb{R}^\ell_+$. That proves part (i) and thus the proof is complete.

This result provides an extension of the classical Non-Substitution Theorem, by assuming that the technology includes an input-output matrix $\mathbf{M}$ with quite sensible properties. It shows that all final demand vectors $\mathbf{d} \in \text{Im}(\mathbf{M}) \cap \mathbb{R}^\ell_+$ can be efficiently realized with this technique. That is, changing the final demand does not require changing the technique in use, as long as these changes are within the subset of $\mathbb{R}^\ell_+$ that is spanned by $\mathbf{M}$. It also proves that the revenues generated by any alternative technique $\mathbf{G}$, when evaluated at the efficiency prices $\mathbf{p}^*$ (those associated with the technique $\mathbf{M}$), are smaller than or equal to the corresponding production costs.

As $n \leq \ell$, this extended version of the Non-Substitution Theorem has only a local nature. That is, it refers to those changes in the demand that occur within the cone $\text{Im}(\mathbf{M}) \cap \mathbb{R}^\ell_+$. The efficiency prices are relative to aggregate net production plans within that set, accordingly. That is, equilibrium prices are uniquely determined as long as the demand remains within the set $\text{Im}(\mathbf{M}) \cap \mathbb{R}^\ell_+$.4

4Needless to say, in the special case in which $n = \ell$ this theorem provides a full non-substitution theorem (that is, $\mathbf{M}$ is an invertible matrix which can produce efficiently all final demand vectors in $\mathbb{R}^\ell_+$ and $\mathbf{p}^*$ is uniquely determined).
This feature illustrates well what may indeed happen in real world economies. Namely, substantial changes in the demand may require changes in the technology and not only in the scale of operations.

The fact that $\text{Im}(M) \cap \mathbb{R}^k_+$ is a convex cone has also some relevant implications. On the one hand, note that if $d \in \text{Im}(M) \cap \mathbb{R}^k_+$ then any vector of the form $\lambda d$ (with $\lambda > 0$) will also be in $\text{Im}(M) \cap \mathbb{R}^k_+$. Therefore, changes in the scale of the demand vector will not require changes in the technique in use. Changes of this nature will occur when endowments vary and consumers’ preferences are homothetic, or when we replicate the consumers in the economy. On the other hand, the size of individual demand changes which do not get out of the cone $\text{Im}(M) \cap \mathbb{R}^k_+$, is related to the size of the economy (e.g. the number of consumers and the size of their endowments). This suggests that large economies are less likely to have to change the technique in use due to changes in tastes or endowments. This may be of some import when the change of technique is costly. In that case our Non-Substitution Theorem points out to the existence of some economies of scale, in spite of constant returns to scale.

References


