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Bounded response and Arrow's theorem

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Bounded response and Arrow's impossibility*

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Abstract

We propose a new axiom called *bounded response*, which says that the smallest change in an agent's preference leads to the smallest or no change in the aggregated preference in the society. This axiom is weaker than *independence of irrelevant alternatives* à la Arrow. We show that *bounded response* together with a weak axiom imply dictatorship whenever there are four or more alternatives. Our result offers a new perspective on Arrow's theorem: neither independence property nor informational efficiency in independence of irrelevant alternatives is necessary for the impossibility. A new technique is employed in the proof.

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1 Introduction

Social welfare functions map agents' preferences to a social preference. We investigate the possibility of constructing "nice" social welfare functions.

Our key axiom is *bounded response*. Social welfare functions satisfy *bounded response* if the smallest change in a preference profile leads to the smallest change, if any, in the social preference. Our main result is that on the universal domain of preferences, *bounded response* and *Pareto efficiency* almost always imply dictatorship. The exception is the three-alternative case. In the three-alternative case, when there are three or more agents, *bounded response* and *Pareto efficiency* do not imply dictatorship, but they imply the existence of a "weak dictator".

Our analysis is significant in at least three aspects. First, we have an impossibility theorem with a new simple axiom. We discuss *bounded response* shortly in this section. Second, our result changes how to read Arrow's theorem (Arrow, 1951, 1963) that would be one of the most famous and important theorems in economics. It can be readily seen that *bounded response* is logically weaker than *independence of irrelevant alternatives*. Because we have the impossibility with *bounded response*, the impossibility of Arrow's theorem is *not* due to "independence property" or "informational efficiency" of *independence of irrelevant alternatives*.[1] Instead, a "side effect" of *independence of irrelevant alternatives*, i.e., bounded response of social preferences to a change of agents' preferences, is sufficient for the impossibility. Third, our proof shows the applicability of topological arguments to discrete models. Although our arguments do not need any knowledge on topology, the reader familiar with algebraic topology would notice that basic concepts and results in the fundamental group theory in algebraic topology are behind our arguments.

Bounded response is justifiable in several ways. First, *bounded response* is logically weaker than *independence of irrelevant alternatives*. Thus, when social welfare functions are required to satisfy *independence of irrelevant alternatives*, they should satisfy *bounded response*. Second, *bounded response* is natural to some extent. We would be surprised if a tiny change of our preferences would lead to a substantial change of a social preference. Third, *bounded response* ensures that social preferences are not affected very much by small errors in reporting preferences. Assume that due to lack of information or false information on

[1]See, for example, Young (1995) for arguments supporting *independence of irrelevant alternatives* from a normative viewpoint:

> "it is desirable to know, for example, that the relative ranking of candidates for political office would not be changed if purely hypothetical candidates were included on the ballot."

Independence of irrelevant alternatives is also considered as an axiom of *informational efficiency* (Suzumura, 2002).

alternatives, agents cannot formulate their preferences "correctly". In such cases, *bounded response* says that small errors in stating preferences do not make a big difference. This is a desirable property from the viewpoint of agents.

To see implications of *bounded response*, we discuss a difference between *bounded response* and *independence of irrelevant alternatives*. *Independence of irrelevant alternatives* says that for each pair of alternatives x and y, a social preference over x and y depends only on agents' preferences over x and y. For simplicity, consider the case where both agents' preferences and social preferences are linear orders.[2] Choose an agent, called agent i, and consider a preference profile $\boldsymbol{R}$ such that agent i ranks x and y consecutively. Let R be the social preference at $\boldsymbol{R}$. Assume that agent i interchanges the positions of x and y. We regard this change as the "smallest change". According to *independence of irrelevant alternatives*, if x and y are not consecutively ranked in R, the social preference cannot respond to this change of agent i's preference, i.e., the new social preference is R. On the other hand, according to *bounded response*, regardless of how x and y are ranked in R, each linear order obtained by interchanging the positions of any one pair of consecutively ranked alternatives in R can be the new social preference. *Bounded response* puts a restriction on how much a social preference can change, while *independence of irrelevant alternatives* puts a restriction on which part of a social preference can or cannot change. In this sense, *bounded response* is distinct from *independence of irrelevant alternatives* and its variants. Especially, note that *bounded response* does not inherit "independence property" or "informational efficiency" from *independence of irrelevant alternatives*.

The proof of our impossibility theorem is quite nontrivial, as we describe as follows. We say that a preference is *adjacent* to another if these are different by the "smallest change," i.e., disagree only between two alternatives. This adjacent relation is illustrated by a graph in which each vertex represents a preference, and each edge represents a pair of adjacent preferences. *Bounded response*, our primary axiom, requires that two adjacent preference profiles lead to two adjacent or the same social preferences. Thus, the property of a social welfare function satisfying *bounded response* can be captured by analyzing geometric characteristics of the images in this graph. Specifically, we consider a *loop* of preferences, which is a finite sequence of preferences such that each preference is adjacent or equal to the next one, and the last preference is adjacent or equal to the initial one. Since *bounded response* implies that iterating the smallest change in preference profiles leads to iteration of the smallest (or no) changes in social preferences, if an agent's preference moves along a loop c while others' preferences are fixed, then the social prefer-

[2] In Section 5, social preferences are formulated as weak orders.

ence also moves along some loop c'. *Adjacency-extended unanimity*, our secondary axiom, requires unanimity together with a condition that if a preference profile consists of two adjacent preferences, then the social preference must be either of the two. This axiom is weaker than *Pareto efficiency*, which is a standard one. We discuss implications derived from *bounded response* and *adjacency-extended unanimity* in two separate cases; the three-alternative case and the four-or-more-alternative case.

First, let us consider the case with three alternatives. In this case, there are six preferences represented by a hexagon-shaped graph. Each loop in this environment is classified by a "winding number" which counts how many times it rotates along the hexagon. Let $\bar{c}$ be a loop which makes a single counterclockwise rotation. Applying *bounded response* and *adjacency-extended unanimity*, we can show that if each agent's preference sequentially moves along $\bar{c}$ (first, agent 1's preference moves along $\bar{c}$, next agent 2's, and so on), then the social preference also moves once along $\bar{c}$. This implies that there exists an agent i such that for each fixed preference profile of the other agents, if i's preference moves along $\bar{c}$, then the social preference moves once along the hexagon, although the social preference may not coincide with i's preference. Since the image of the social preferences covers the entire set of preferences, we call such an agent a *manipulator* who can manipulate the social preference if the others' preferences are known. By *adjacency-extended unanimity*, we can show that such a manipulator is unique. Moreover, we can show that the social preference equals the manipulator's preference if the preference profile of the others consists of a single preference or two adjacent ones. These results suggest a "near" impossibility in the three-alternative case.

Next, let us consider the case with m (≥ 4) alternatives, to which the results in the three-alternative case cannot be carried over directly because *bounded response* deviates from the independence property. If a preference $\bar{R}$ and an integer t ($1 \leq t \leq m-2$) are fixed, exchanging the ranks of the tth, $(t+1)$th, and $(t+2)$th alternatives in $\bar{R}$ yields a hexagon-shaped graph as in the three-alternative case. Then, the argument in the previous paragraph leads to the same conclusion. In particular, under *bounded response* and *adjacency-extended unanimity*, for each t, if i's preference moves along the hexagon with t, then the social preference moves once along the hexagon with t. For each s such that $1 \leq s \leq m-3$, the hexagon when $t = s$ and the one when $t = s+1$ share an edge, i.e., both include two preferences $\bar{R}$, and $\bar{R}'$ defined as a preference in which the ranks of the $(s+1)$th and $(s+2)$th alternatives in $\bar{R}$ are exchanged. Since moving from $\bar{R}$ to $\bar{R}'$ is a part of a rotation when $t = s$ and when $t = s+1$, the social preferences corresponding to i's preference moving from $\bar{R}$ to $\bar{R}'$ must fall within intersection of the hexagons when $t = s$ and when $t = s+1$. This implies that, by *adjacency-extended unanimity*, the social preference must equal the manipulator's preference. Thus, *bounded response* and *adjacency-extended unanimity* imply dictatorship whenever there are four or more alternatives.

We discuss the relationship of this paper to the literature. After the seminal work of Arrow (1951, 1963), many papers have improved the proof and relaxed the axioms.[3] For example, Barberà (1980), Blackorby et al. (1984), Reny (2001), Eliaz (2004), Geanakoplos (2005), Cato (2010), Man and Takayama (2013), and many others prove Arrow's theorem in various ways. Since, as far as we know, all of them heavily rely on the "independence property" of *independence of irrelevant alternatives*, their techniques of the proofs cannot be applied to *bounded response* under which the social preference over x and y may be reversed when each agent's preference over x and y remains the same. As we already mentioned, our proof is reminiscent of techniques in topological social choice. See Baigent (2010) for a survey of topological social choice theory. Baryshnikov (1993) proves Arrow's theorem by topological methods. Tanaka (2006, 2009) discuss relations of Arrow's theorem to Brouwer's fixed point theorem also by topological methods. We note that the proofs in these three papers crucially depend on *independence of irrelevant alternatives*, and hence distinct from ours. We also note that unlike these papers, our argument does not require knowledge of topology.

Various weaker axioms than *independence of irrelevant alternatives* have been proposed by many papers including Blau (1971), Hansson (1973), Baigent (1987), Young (1988), Campbell and Kelly (2000), Yu (2014) among others. To the best of our knowledge, they inherit "independence property" of *independence of irrelevant alternatives*, and are conceptually different from *bounded response*. Also, none of them are logically weaker than *bounded response*. For example, *independence of some alternative* by Campbell and Kelly (2000) says that a social preference over x and y depends on agents' preferences over some proper subset of the set of all alternatives. Sato (2015) considers *bounded response*, but he uses it to see a relationship between *nonmanipulability* and *independence of irrelevant alternatives* of social welfare functions, and his analysis is different from ours very much.

The rest of the paper is organized as follows. In Section 2, we introduce basic notations, the axioms, and the concepts that will play an important role in the proofs. In Section 3, we prove a "near" impossibility result in the case with three alternatives, and prepare for Section 4 in which our main impossibility result is shown when the society has four or more alternatives. Section 5 extends the main result to the case with the social preferences that may contain ties, and Section 6 concludes.

[3]See Campbell and Kelly (2002) for a survey.

2 Definitions

A society consists of n agents in $N = \{1, \ldots, n\}$, and has m feasible alternatives in X. Let $\mathcal{L}$ be the set of all *preference relations*, namely orders on X which are complete, transitive, and antisymmetric. We denote typically by $\boldsymbol{R} \in \mathcal{L}^n$ a preference profile of n agents, and by $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$ a preference profile of $n-1$ agents except agent $i \in N$. Given $\boldsymbol{R} \in \mathcal{L}^n$, we denote by R_i the preference of agent i in $\boldsymbol{R}$. A function $f : \mathcal{L}^N \to \mathcal{L}$ is a *social welfare function* on $\mathcal{L}$. An agent $i \in N$ is a *dictator* if $f(R_i, \boldsymbol{R}_{-i}) = R_i$ for each $R_i \in \mathcal{L}$ and each $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$. The social welfare function is *dictatorial* if there exists a dictator. An agent $i \in N$ is a *manipulator* if for each $R \in \mathcal{L}$ and each $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$, there exists $\hat{R}_i \in \mathcal{L}$ such that $f(\hat{R}_i, \boldsymbol{R}_{-i}) = R$. A manipulator can achieve each social preference $R \in \mathcal{L}$ by reporting an appropriate preference $\hat{R}_i \in \mathcal{L}$ depending upon the others' $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$. However, R is not necessarily equal to $\hat{R}_i$. Thus, a dictator is a manipulator, but not vice versa.

For each $R_i \in \mathcal{L}$, we denote the kth ranked alternative according to R_i by $r^k(R_i)$. Two preference relations $R_i, R_i' \in \mathcal{L}$ are *adjacent* if $R_i \neq R_i'$, and there exist two alternatives $x, y \in X$ such that for each pair of alternatives $z, w \in X$ with $\{z, w\} \neq \{x, y\}$, $z \; R_i \; w$ if and only if $z \; R_i' \; w$. We note that R_i and R_i' are adjacent if and only if R_i' can be obtained by reversing the order of one pair of alternatives consecutively ranked in R_i. Two preference profiles $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{L}^N$ are *adjacent* if there is $i \in N$ such that R_i and R_i' are adjacent and $\boldsymbol{R}_{-i} = \boldsymbol{R}'_{-i}$.

2.1 Axioms

We introduce three axioms of a social welfare function f:

Axiom 1 (*Bounded response*)**.** If two preference profiles $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{L}^n$ are adjacent, then $f(\boldsymbol{R})$ and $f(\boldsymbol{R}')$ are either adjacent or the same.

This is our main axiom, which requires that the smallest change in an agent's preference should result in the smallest or no change in the social preference. A social welfare function f satisfies *independence of irrelevant alternatives* if for each $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{L}^n$ and each $x, y \in X$, if $\boldsymbol{R}$ and $\boldsymbol{R}'$ agree on $\{x, y\}$, then $f(\boldsymbol{R})$ and $f(\boldsymbol{R}')$ agree on $\{x, y\}$. We note that *bounded response* is weaker than *independence of irrelevant alternatives* which implies that when two alternatives x and y are consecutively ranked, exchanging the ranks of x and y affects only the ranks of x and y in the social preference.

Proposition 1. Independence of irrelevant alternatives *implies* bounded response.

Proof. Fix an agent $i \in N$, a preference profile $\boldsymbol{R} \in \mathcal{L}^n$, and a preference $R'_i \in \mathcal{L}$ which is adjacent to R_i, arbitrarily. Let $x, y \in X$ be two alternatives such that $x \; R_i \; y$ and $y \; R'_i \; x$. Since R_i and R'_i are adjacent, for each $z, w \in X$ such that $\{z, w\} \neq \{x, y\}$, $z \; R_i \; w$ if and only if $z \; R'_i \; w$. Two preference profiles $\boldsymbol{R}$ and $(R'_i, \boldsymbol{R}_{-i})$ coincide over each pair $z, w \in X$ such that $\{z, w\} \neq \{x, y\}$. Then, *independence of irrelevant alternatives* implies that the social preferences $f(\boldsymbol{R})$ and $f(R'_i, \boldsymbol{R}_{-i})$ coincide over each pair $z, w \in X$ such that $\{z, w\} \neq \{x, y\}$. Hence, $f(\boldsymbol{R})$ and $f(R'_i, \boldsymbol{R}_{-i})$ are adjacent or the same. Since the choices of i, $\boldsymbol{R}$, and R'_i were arbitrary, we have proved *bounded response*. $\square$

Bounded response is weaker than *independence of irrelevant alternatives* because when the ranks of x and y are exchanged between R_i and R'_i, *bounded response* allows for social preferences such that the ranks of z and w are exchanged between $f(\boldsymbol{R})$ and $f(R'_i, \boldsymbol{R}_{-i})$ where $\{z, w\} \neq \{x, y\}$.

Axiom 2 (*Adjacency-extended unanimity*)**.** For each $\boldsymbol{R} \in \mathcal{L}^n$, if $\{R_i \mid i \in N\} = \{R\}$ for some $R \in \mathcal{L}$, then $f(\boldsymbol{R}) = R$, and if $\{R_i \mid i \in N\} = \{R, R'\}$ for some adjacent $R, R' \in \mathcal{L}$, then $f(\boldsymbol{R}) \in \{R, R'\}$.

This axiom is similar to *unanimity* which requires $f(\boldsymbol{R}) = R$ for each $\boldsymbol{R} = (R, R, \ldots, R) \in \mathcal{L}^n$, but stronger than *unanimity* in that when $\boldsymbol{R} \in \mathcal{L}^n$ consists of only two adjacent preferences, $f(\boldsymbol{R})$ must be either of the two.

Axiom 3 (*Pareto efficiency*)**.** For each pair of alternatives $x, y \in X$ and each preference profile $\boldsymbol{R} \in \mathcal{L}^n$, if $x \; R_i \; y$ for each $i \in N$, then $x \; f(\boldsymbol{R}) \; y$.

This axiom implies that for each $x, y \in X$, if each agent prefers x to y, then x must be preferred to y in the social preference. We note that *Pareto efficiency* is stronger than *adjacency-extended unanimity*; If $\{R_i \mid i \in N\}$ consists of a single preference R, then *Pareto efficiency* implies $f(\boldsymbol{R}) = R$. Suppose that $\{R_i \mid i \in N\}$ consists of one pair of adjacent preferences R and R' such that they disagree on x and y. Then, *Pareto efficiency* implies that for each $z, w \in X$ such that $\{z, w\} \neq \{x, y\}$, the social preference and the agents' preferences agree on z and w. The social preference on x and y agrees with either R or R'. In the former case, $f(\boldsymbol{R}) = R$, and in the latter case, $f(\boldsymbol{R}) = R'$.

2.2 Loops of preferences

For an integer $K \geq 1$, a sequence of preferences $c = (R^1, \ldots, R^K) \in \mathcal{L}^K$ is a *loop* or a *cycle of preferences* (or simply a *loop*) of length K if R^k and R^{k+1} are either adjacent or the same for each k with $1 \leq k \leq K$ (where we regard $R^{K+1} = R^1$ by rule). Let C be the set of all loops.

We denote the length of $c \in C$ by length(c), and the kth preference by $c(k)$ for each k ($1 \leq k \leq \text{length}(c)$). As an abuse of notation, if $k' > \text{length}(c)$ or $k' < 1$, $c(k')$ is defined as $c(k') = c(k)$ where k is an integer satisfying $1 \leq k \leq \text{length}(c)$ and $k' = p \cdot \text{length}(c) + k$ for some integer p.

We introduce an equivalence relation $\simeq$ in C, which may be interpreted as "can be continuously deformed to".

Definition 1. Let $c, c' \in C$ be two loops. We say that c and c' are *connected* with each other and denote $c \simeq c'$ if there exists a finite sequence of loops $c = c_0, c_1, c_2, \ldots, c_{L-1}, c_L = c'$ such that for each l ($1 \leq l \leq L$), c_{l-1} and c_l satisfy either of the following:

(1) $\text{length}(c_l) = \text{length}(c_{l-1}) + 1$, and there is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(c_{l-1})$) such that

$$c_l(k) = \begin{cases} c_{l-1}(k) & \text{if } 1 \leq k \leq \bar{k}, \\ c_{l-1}(k-1) & \text{if } \bar{k}+1 \leq k \leq \text{length}(c_l). \end{cases}$$

(2) $\text{length}(c_{l-1}) = \text{length}(c_l) + 1$, and there is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(c_l)$) such that

$$c_{l-1}(k) = \begin{cases} c_l(k) & \text{if } 1 \leq k \leq \bar{k}, \\ c_l(k-1) & \text{if } \bar{k}+1 \leq k \leq \text{length}(c_{l-1}). \end{cases}$$

(3) $\text{length}(c_l) = \text{length}(c_{l-1})$, and there is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(c_{l-1})$) such that $c_{l-1}(k) = c_l(k)$ whenever $k \neq \bar{k}$.

If the first condition holds, the loop c_{l-1} is extended to c_l by replicating the $\bar{k}$th preference in c_{l-1}. This is a trivial way of extending the length of a loop. The second condition imposes a converse way, shortening the loop. The third condition introduces a continuous transformation of a loop; since $c_l(\bar{k}-1) = c_{l-1}(\bar{k}-1)$ and $c_l(\bar{k}+1) = c_{l-1}(\bar{k}+1)$, both $c_{l-1}(\bar{k})$ and $c_l(\bar{k})$ must be adjacent or equal to $c_{l-1}(\bar{k}-1)$ and $c_{l-1}(\bar{k}+1)$. Thus this is a replacement of the $\bar{k}$th preference that maintains the adjacent relations in the loops. We note that $\simeq$ is an equivalence relation.

An easy observation shows that it does not matter which preference is the initial in a loop.

Lemma 2. *For each $c \in C$ and each integer p ($1 \leq p \leq \text{length}(c)$), $c \simeq \big(c(1+p), c(2+p), \ldots, c(\text{length}(c)+p)\big)$.*

Proof. Let $c = (R^1, R^2, R^3, \dots, R^K)$. Then

$$\begin{aligned} c &\simeq (R^1, R^2, R^3, \dots, R^K, R^K) && \text{(by (1) in Definition 1)} \\ &\simeq (R^1, R^2, R^3, \dots, R^K, R^1) && \text{(by (3) in Definition 1)} \\ &\simeq (R^2, R^2, R^3, \dots, R^K, R^1) && \text{(by (3) in Definition 1)} \\ &\simeq (R^2, R^3, \dots, R^K, R^1) && \text{(by (2) in Definition 1).} \end{aligned}$$

Therefore the statement is shown with $p = 1$. The general case is proved by induction. □

Analogously, for an integer $K \geq 1$, a sequence of preference profiles $\gamma = (\boldsymbol{R}^1, \dots, \boldsymbol{R}^K) \in (\mathcal{L}^n)^K$ is a *loop of preference profiles* of length K if $\boldsymbol{R}^k$ and $\boldsymbol{R}^{k+1}$ are either adjacent or the same for each k with $1 \leq k \leq K$ (where we regard $\boldsymbol{R}^{K+1} = \boldsymbol{R}^1$ by rule). Let Γ be the set of all loops of preference profiles. We denote the length of $\gamma \in \Gamma$ by $\text{length}(\gamma)$, and the kth preference profile by $\gamma(k)$ for each k ($1 \leq k \leq \text{length}(\gamma)$). As an abuse of notation, if $k' > \text{length}(\gamma)$ or $k' < 1$, $\gamma(k')$ is defined as $\gamma(k') = \gamma(k)$ where k is an integer satisfying $1 \leq k \leq \text{length}(\gamma)$ and $k' = p \cdot \text{length}(\gamma) + k$ for some integer p.

We introduce an equivalence relation $\simeq$ in Γ.

Definition 2. Let $\gamma, \gamma' \in \Gamma$ be two loops of preference profiles. We say that γ and γ' are *connected* with each other and denote $\gamma \simeq \gamma'$ if there exists a finite sequence of loops of preference profiles $\gamma = \gamma_0, \gamma_1, \gamma_2, \dots, \gamma_{L-1}, \gamma_L = \gamma'$ such that for each l ($1 \leq l \leq L$), γ_{l-1} and γ_l satisfy either of the following:

(1) $\text{length}(\gamma_l) = \text{length}(\gamma_{l-1}) + 1$, and there is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(\gamma_{l-1})$) such that

$$\gamma_l(k) = \begin{cases} \gamma_{l-1}(k) & \text{if } 1 \leq k \leq \bar{k}, \\ \gamma_{l-1}(k-1) & \text{if } \bar{k}+1 \leq k \leq \text{length}(\gamma_l). \end{cases}$$

(2) $\text{length}(\gamma_{l-1}) = \text{length}(\gamma_l) + 1$, and there is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(\gamma_l)$) such that

$$\gamma_{l-1}(k) = \begin{cases} \gamma_l(k) & \text{if } 1 \leq k \leq \bar{k}, \\ \gamma_l(k-1) & \text{if } \bar{k}+1 \leq k \leq \text{length}(\gamma_{l-1}). \end{cases}$$

(3) $\text{length}(\gamma_l) = \text{length}(\gamma_{l-1})$, and there is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(\gamma_{l-1})$) such that $\gamma_{l-1}(k) = \gamma_l(k)$ whenever $k \neq \bar{k}$.

We introduce another kind of manipulation of the loop of preference profiles in which two preference profiles are removed.

Lemma 3. *For a loop of preference profiles $\gamma = (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^K) \in \Gamma$ with* $\text{length}(\gamma) \geq 3$, *suppose that there exists $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(\gamma)$) such that $\boldsymbol{R}^{\bar{k}}$ and $\boldsymbol{R}^{\bar{k}+3}$ are adjacent or the same. Then γ is connected to the loop given by removing $\boldsymbol{R}^{\bar{k}+1}$ and $\boldsymbol{R}^{\bar{k}+2}$ from γ, i.e.*

$$\gamma \simeq (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{\bar{k}-1}, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}+3}, \boldsymbol{R}^{\bar{k}+4}, \ldots, \boldsymbol{R}^K).$$

Proof.

$$\begin{aligned}
\gamma &= (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}+1}, \boldsymbol{R}^{\bar{k}+2}, \boldsymbol{R}^{\bar{k}+3}, \ldots, \boldsymbol{R}^K) \\
&\simeq (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}+1}, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}+3}, \ldots, \boldsymbol{R}^K) && \text{(by (3) in Definition 2)} \\
&\simeq (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}+3}, \ldots, \boldsymbol{R}^K) && \text{(by (3) in Definition 2)} \\
&\simeq (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{\bar{k}}, \boldsymbol{R}^{\bar{k}+3}, \ldots, \boldsymbol{R}^K) && \text{(by (2) in Definition 2).}
\end{aligned}$$

□

Suppose that two loops $c, c' \in C$ satisfy $c(1) = c'(1)$. Then we define a concatenation of c and c' denoted by $c \cdot c' \in C$ as

$$(c \cdot c')(k) = \begin{cases} c(k) & \text{if } 1 \leq k \leq \text{length}(c), \\ c'(k - \text{length}(c)) & \text{if } \text{length}(c) + 1 \leq k \leq \text{length}(c) + \text{length}(c'). \end{cases}$$

Note that $\text{length}(c \cdot c') = \text{length}(c) + \text{length}(c')$. For a loop $c \in C$, let $c^{-1} \in C$ be the inverse of c, i.e., $c^{-1}(k) = c(\text{length}(c) + 2 - k)$. For a loop $c \in C$ and a positive integer p, $c^p := \overbrace{c \cdot c \cdot \cdots \cdot c}^{p}$, and $c^{-p} := \overbrace{c^{-1} \cdot c^{-1} \cdot \cdots \cdot c^{-1}}^{p}$. By rule we define c^0 as the loop $(c(1))$ of length 1. The inverse γ^{-1} of a loop of preference profiles, the concatenation of two loops of preference profiles $\gamma \cdot \gamma'$, and γ^p are defined analogously.

For a social welfare function f, and a loop of preference profiles $\gamma = (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^K) \in \Gamma$, we denote $f(\gamma) = \big(f(\boldsymbol{R}^1), \ldots, f(\boldsymbol{R}^K)\big) \in \mathcal{L}^K$ as a slight abuse of notations. Given a social welfare function satisfying *bounded response*, the following lemma is a straight-forward observation:

Lemma 4. *Suppose that a social welfare function f satisfies* bounded response. *Then*

(a) for each $\gamma \in \Gamma$, $f(\gamma)$ is a loop, and

(b) for each $\gamma, \gamma' \in \Gamma$, $\gamma \simeq \gamma'$ implies $f(\gamma) \simeq f(\gamma')$.

$\bar{R}^1$	$\bar{R}^2$	$\bar{R}^3$	$\bar{R}^4$	$\bar{R}^5$	$\bar{R}^6$
x^1	x^1	x^3	x^3	x^2	x^2
x^2	x^3	x^1	x^2	x^3	x^1
x^3	x^2	x^2	x^1	x^1	x^3

Table 1: Preferences

Proof. (a) Since γ is a loop of preference profiles, for each k ($1 \leq k \leq \text{length}(\gamma)$), γ_k and γ_{k+1} are either adjacent or the same. If γ_k and γ_{k+1} are the same, then $f(\gamma_k)$ and $f(\gamma_{k+1})$ are the same loop. If γ_k and γ_{k+1} are adjacent, *bounded response* implies that $f(\gamma_k)$ and $f(\gamma_{k+1})$ are either adjacent or the same. Therefore in both cases, $f(\gamma_k)$ and $f(\gamma_{k+1})$ are either adjacent or the same. Hence $f(\gamma)$ is a loop.

(b) Suppose that $\gamma \simeq \gamma'$. Then there exists a sequence $\gamma = \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{L-1}, \gamma_L = \gamma'$ that satisfies either of the conditions in Definition 2 for each l ($1 \leq l \leq L$). This immediately implies that the sequence of loops $f(\gamma) = f(\gamma_0), f(\gamma_1), f(\gamma_2), \ldots, f(\gamma_{L-1}), f(\gamma_L) = f(\gamma')$ satisfies either of the conditions in Definition 1 for each l ($1 \leq l \leq L$). Hence $f(\gamma) \simeq f(\gamma')$. □

By definition, we have $f(\gamma \cdot \gamma') = f(\gamma) \cdot f(\gamma')$ for each $\gamma, \gamma' \in \Gamma$ such that $\gamma(1) = \gamma'(1)$. Note that Lemma 4 guarantees $f(\gamma), f(\gamma') \in C$ and $\big(f(\gamma)\big)(1) = \big(f(\gamma')\big)(1)$, and thus the concatenation is well-defined.

3 Three alternatives

In this section, we focus on the case with $|X| = 3$. Let $X = \{x^1, x^2, x^3\}$. There are six preferences $\bar{R}^1, \ldots, \bar{R}^6$ in $\mathcal{L}$ as presented in Table 1, in which each column defines a preference at the top row. For example, $\bar{R}^1$ is a preference satisfying $x^1 \ \bar{R}^1 \ x^2 \ \bar{R}^1 \ x^3$, and so on. By rule, let $\bar{R}^{l'} = \bar{R}^l$ if $l' = 6p + l$ for some integer p. In this notation, R and R' are adjacent if and only if there exists an integer l, l' such that $R = \bar{R}^l$, $R' = \bar{R}^{l'}$ and $l - l' = 6p + 1$ or $6p - 1$ with some integer p. Figure 1 presents a graph in which a vertex represents a preference, and two vertices are connected by an edge if and only if the corresponding two preferences are adjacent. For each loop $c \in C$ and each k ($1 \leq k \leq 6$), if $c(k) = \bar{R}^l$, then $c(k+1)$ equals either $\bar{R}^{l-1}$, $\bar{R}^l$, or $\bar{R}^{l+1}$ by the definition of a loop. If $\bar{R}^l$ and $\bar{R}^{l'}$ are

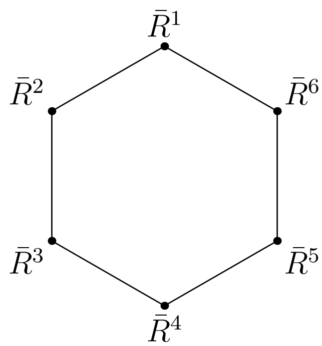

Figure 1: A graph representing adjacent relations.

adjacent or the same, we define an integer $\alpha(\bar{R}^l, \bar{R}^{l'})$ by

$$\alpha(\bar{R}^l, \bar{R}^{l'}) := \begin{cases} 1 & \text{if } \bar{R}^{l'} = \bar{R}^{l+1}, \\ 0 & \text{if } \bar{R}^{l'} = \bar{R}^{l}, \\ -1 & \text{if } \bar{R}^{l'} = \bar{R}^{l-1}. \end{cases}$$

We introduce a useful function w from C to the set of rational numbers:[4] Let

$$w(c) = \frac{1}{6} \sum_{k=1}^{\text{length}(c)} \alpha\big(c(k), c(k+1)\big).$$

By definition, $w(c \cdot c') = w(c) + w(c')$ whenever $c \cdot c'$ is well-defined.

For each l ($1 \leq l \leq 6$), let $\bar{c}_{\bar{R}^l} \in C$ be the loop $(\bar{R}^l, \bar{R}^{l+1}, \ldots, \bar{R}^{l+5})$. For example, we

[4]We will later prove that $w(c)$ is an integer for each loop $c \in C$.

have

$$
\begin{aligned}
w(\bar{c}_{\bar{R}^1}) &= w(\bar{R}^1, \bar{R}^2, \bar{R}^3, \bar{R}^4, \bar{R}^5, \bar{R}^6) \\
&= \alpha(\bar{R}^1, \bar{R}^2) + \alpha(\bar{R}^2, \bar{R}^3) + \alpha(\bar{R}^3, \bar{R}^4) + \alpha(\bar{R}^4, \bar{R}^5) + \alpha(\bar{R}^5, \bar{R}^6) + \alpha(\bar{R}^6, \bar{R}^1) \\
&= \frac{1}{6}(1+1+1+1+1+1) = 1, \\
w(\bar{R}^1, \bar{R}^2, &\bar{R}^3, \bar{R}^4, \bar{R}^3, \bar{R}^2) \\
&= \alpha(\bar{R}^1, \bar{R}^2) + \alpha(\bar{R}^2, \bar{R}^3) + \alpha(\bar{R}^3, \bar{R}^4) + \alpha(\bar{R}^4, \bar{R}^3) + \alpha(\bar{R}^3, \bar{R}^2) + \alpha(\bar{R}^2, \bar{R}^1) \\
&= \frac{1}{6}(1+1+1-1-1-1) = 0, \\
w(\bar{c}_{\bar{R}^1}^{-1}) &= w(\bar{R}^1, \bar{R}^6, \bar{R}^5, \bar{R}^4, \bar{R}^3, \bar{R}^2) \\
&= \alpha(\bar{R}^1, \bar{R}^6) + \alpha(\bar{R}^6, \bar{R}^5) + \alpha(\bar{R}^5, \bar{R}^4) + \alpha(\bar{R}^4, \bar{R}^3) + \alpha(\bar{R}^3, \bar{R}^2) + \alpha(\bar{R}^2, \bar{R}^1) \\
&= \frac{1}{6}(-1-1-1-1-1-1) = -1.
\end{aligned}
$$

Intuitively, $w(c)$ counts how many times the loop c rotates counterclockwise along the hexagon illustrated in Figure 1. The following lemma reveals why the function w is useful in this environment:

Lemma 5. *(a) For each loop $c \in C$, $w(c)$ is an integer.*

(b) For each $c, c' \in C$, $c \simeq c'$ if and only if $w(c) = w(c')$.

(c) For each $c \in C$, $c \simeq \bar{c}_{\bar{R}^1}^{w(c)}$.

The proof is given in Appendix A.1. As a corollary, we obtain the following lemma:

Lemma 6. *Suppose that $c \in C$ is a loop of length 6. Then, either $c \simeq (c(1)) \in C$, $c = \bar{c}_{c(1)}$, or $c = (\bar{c}_{c(1)})^{-1}$.*

Although the proof of Lemma 12 (a) in Section 4 is directly applicable to this lemma, we provide a short proof:

Proof. Suppose that $c \not\simeq (c(1))$. Then, Lemma 5 (c) implies $w(c) \neq 0$. By Lemma 5 (a), $w(c) = 1$ or -1. If $w(c) = 1$, then $\alpha\big(c(k), c(k+1)\big) = 1$ for all k. The unique loop satisfying such a condition is obviously $c = \bar{c}_{c(1)}$. If $w(c) = 1$, then $\alpha\big(c(k), c(k+1)\big) = -1$ for all k. The unique loop satisfying such a condition is obviously $c = (\bar{c}_{c(1)})^{-1}$. □

We consider the following loop of preference profiles $\bar{\gamma} \in \Gamma$ with length $6n$: For each

p ($1 \leq p \leq 6$) and each q ($1 \leq q \leq n$),

$$(\bar{\gamma}((p-1)n+q))_i = \begin{cases} \bar{R}^{p+1} & \text{if } 1 \leq i \leq q-1, \\ \bar{R}^p & \text{if } q \leq i \leq n. \end{cases}$$

That is,

$$\begin{aligned}\bar{\gamma} = \big(&(\bar{R}^1, \bar{R}^1, \bar{R}^1, \ldots, \bar{R}^1, \bar{R}^1, \bar{R}^1), (\bar{R}^2, \bar{R}^1, \bar{R}^1, \ldots, \bar{R}^1, \bar{R}^1, \bar{R}^1), (\bar{R}^2, \bar{R}^2, \bar{R}^1, \ldots, \bar{R}^1, \bar{R}^1, \bar{R}^1),\\ &\ldots, (\bar{R}^2, \bar{R}^2, \bar{R}^2, \ldots, \bar{R}^2, \bar{R}^1, \bar{R}^1), (\bar{R}^2, \bar{R}^2, \bar{R}^2, \ldots, \bar{R}^2, \bar{R}^2, \bar{R}^1),\\ &(\bar{R}^2, \bar{R}^2, \bar{R}^2, \ldots, \bar{R}^2, \bar{R}^2, \bar{R}^2), (\bar{R}^3, \bar{R}^2, \bar{R}^2, \ldots, \bar{R}^2, \bar{R}^2, \bar{R}^2), (\bar{R}^3, \bar{R}^3, \bar{R}^2, \ldots, \bar{R}^2, \bar{R}^2, \bar{R}^2),\\ &\ldots, (\bar{R}^3, \bar{R}^3, \bar{R}^3, \ldots, \bar{R}^3, \bar{R}^2, \bar{R}^2), (\bar{R}^3, \bar{R}^3, \bar{R}^3, \ldots, \bar{R}^3, \bar{R}^3, \bar{R}^2),\\ &\ldots,\\ &(\bar{R}^6, \bar{R}^6, \bar{R}^6, \ldots, \bar{R}^6, \bar{R}^6, \bar{R}^6), (\bar{R}^1, \bar{R}^6, \bar{R}^6, \ldots, \bar{R}^6, \bar{R}^6, \bar{R}^6), (\bar{R}^1, \bar{R}^1, \bar{R}^6, \ldots, \bar{R}^6, \bar{R}^6, \bar{R}^6),\\ &\ldots, (\bar{R}^1, \bar{R}^1, \bar{R}^1, \ldots, \bar{R}^1, \bar{R}^6, \bar{R}^6), (\bar{R}^1, \bar{R}^1, \bar{R}^1, \ldots, \bar{R}^1, \bar{R}^1, \bar{R}^6)\big).\end{aligned}$$

Further, for each preference profile $\boldsymbol{R} = (\bar{R}^{l_1}, \ldots, \bar{R}^{l_n}) \in \mathcal{L}^n$ and each $i \in N$, let $\gamma_i^{\boldsymbol{R}} \in \Gamma$ be a loop of preference profiles of length 6 defined as

$$(\gamma_i^{\boldsymbol{R}}(k))_j = \begin{cases} \bar{R}^{l_i+k-1} & \text{if } j = i, \\ \bar{R}^{l_j} & \text{if } j \neq i. \end{cases}$$

That is, $\gamma_i^{\boldsymbol{R}}$ is a loop of preference profiles in which agent i's coordinates consist of a loop of preferences $(\bar{R}^{l_i}, \bar{R}^{l_i+1}, \ldots, \bar{R}^{l_i+5})$, and any other agent j's coordinates are constant. In the following lemma, we show that the choice of the initial preference profile $\boldsymbol{R}$ is not important:

Lemma 7. *For each $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{L}^n$ and each $i \in N$, $\gamma_i^{\boldsymbol{R}} \simeq \gamma_i^{\boldsymbol{R}'}$. Further, $\gamma_1^{\boldsymbol{R}} \cdot \gamma_2^{\boldsymbol{R}} \cdot \cdots \cdot \gamma_n^{\boldsymbol{R}} \simeq \gamma_1^{\boldsymbol{R}'} \cdot \gamma_2^{\boldsymbol{R}'} \cdot \cdots \cdot \gamma_n^{\boldsymbol{R}'}$.*

Proof. By transitivity of $\simeq$, it suffices to show the lemma when $\boldsymbol{R}$ and $\boldsymbol{R}'$ are adjacent. Let $i \in N$, and $j \in N$ be such that $R_j \neq R'_j$. If $i = j$, then an argument similar to Lemma 1 shows $\gamma_i^{\boldsymbol{R}} \simeq \gamma_i^{\boldsymbol{R}'}$.

Assume $i \neq j$, and let $R_i = \bar{R}^l$. By Lemma 3,

$$
\begin{aligned}
\gamma_i^{\boldsymbol{R}} &= \big((\bar{R}^l, R_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+1}, R_j, \boldsymbol{R}_{-(i,j)}), \dots, (\bar{R}^{l+4}, R_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+5}, R_j, \boldsymbol{R}_{-(i,j)})\big) \\
&\simeq \big((\bar{R}^l, R_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^l, R'_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+1}, R'_j, \boldsymbol{R}_{-(i,j)}), \\
&\qquad (\bar{R}^{l+1}, R_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+1}, R'_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+2}, R'_j, \boldsymbol{R}_{-(i,j)}), \\
&\qquad \dots, \\
&\qquad (\bar{R}^{l+4}, R_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+4}, R'_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+5}, R'_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+5}, R_j, \boldsymbol{R}_{-(i,j)})\big) \\
&\simeq \big(\gamma_i^{\boldsymbol{R}}(1), (\bar{R}^l, R'_j, \boldsymbol{R}_{-(i,j)}), (\bar{R}^{l+1}, R'_j, \boldsymbol{R}_{-(i,j)}), \dots, (\bar{R}^{l+5}, R'_j, \boldsymbol{R}_{-(i,j)}), \gamma_i^{\boldsymbol{R}}(6)\big) \\
&= \big(\gamma_i^{\boldsymbol{R}}(1), \gamma_i^{\boldsymbol{R}'}, \gamma_i^{\boldsymbol{R}}(6)\big). \qquad (1)
\end{aligned}
$$

Since Lemma 3 implies $\big(\gamma_i^{\boldsymbol{R}}(1), \gamma_i^{\boldsymbol{R}'}, \gamma_i^{\boldsymbol{R}}(6)\big) \simeq \gamma_i^{\boldsymbol{R}'}$, we have $\gamma_i^{\boldsymbol{R}} \simeq \gamma_i^{\boldsymbol{R}'}$.

Also, it can be seen that $\gamma_j^{\boldsymbol{R}} \simeq \big(\gamma_j^{\boldsymbol{R}}(1), \gamma_j^{\boldsymbol{R}'}, \gamma_j^{\boldsymbol{R}}(6)\big)$. In the above process (1) of deformation, the first term remains the same. Therefore, we can apply this deformation without violating each concatenation;

$$
\begin{aligned}
&\gamma_1^{\boldsymbol{R}} \cdot \gamma_2^{\boldsymbol{R}} \cdot \dots \cdot \gamma_{j-1}^{\boldsymbol{R}} \cdot \gamma_j^{\boldsymbol{R}} \cdot \gamma_{j+1}^{\boldsymbol{R}} \cdot \dots \cdot \gamma_n^{\boldsymbol{R}} \\
&\simeq \big(\gamma_1^{\boldsymbol{R}}(1), \gamma_1^{\boldsymbol{R}'}, \gamma_1^{\boldsymbol{R}}(6)\big) \cdot \big(\gamma_2^{\boldsymbol{R}}(1), \gamma_2^{\boldsymbol{R}'}, \gamma_2^{\boldsymbol{R}}(6)\big) \cdot \dots \cdot \big(\gamma_{j-1}^{\boldsymbol{R}}(1), \gamma_{j-1}^{\boldsymbol{R}'}, \gamma_{j-1}^{\boldsymbol{R}}(6)\big) \\
&\qquad \cdot \big(\gamma_j^{\boldsymbol{R}}(1), \gamma_j^{\boldsymbol{R}'}, \gamma_j^{\boldsymbol{R}}(6)\big) \cdot \big(\gamma_{j+1}^{\boldsymbol{R}}(1), \gamma_{j+1}^{\boldsymbol{R}'}, \gamma_{j+1}^{\boldsymbol{R}}(6)\big) \cdot \dots \cdot \big(\gamma_n^{\boldsymbol{R}}(1), \gamma_n^{\boldsymbol{R}'}, \gamma_n^{\boldsymbol{R}}(6)\big) \\
&\simeq \gamma_1^{\boldsymbol{R}'} \cdot \gamma_2^{\boldsymbol{R}'} \cdot \dots \cdot \gamma_n^{\boldsymbol{R}'}. \qquad \text{(by Lemma 2)}
\end{aligned}
$$

□

These loops of preferences satisfy the following relation:

Lemma 8. *For each* $\boldsymbol{R} \in \mathcal{L}^n$, $\bar{\gamma} \simeq \gamma_1^{\boldsymbol{R}} \cdot \gamma_2^{\boldsymbol{R}} \cdot \dots \cdot \gamma_n^{\boldsymbol{R}}$.

The proof is provided in Appendix A.2.

Suppose that a social welfare function f satisfies *bounded response* and *adjacency-extended unanimity*. By *adjacency-extended unanimity*, for each $p = 1, \dots, 6$, $f(\bar{\gamma})((p-1)n+1) = f(\bar{R}^p, \dots, \bar{R}^p) = \bar{R}^p$, and for each $p = 1, \dots, 6$ and each $q = 2, \dots, n$, $f(\bar{\gamma})((p-1)n+q)$ is either $\bar{R}^p$ or $\bar{R}^{p+1}$. Therefore

$$f(\bar{\gamma}) \simeq (\bar{R}^1, \bar{R}^2, \bar{R}^3, \bar{R}^4, \bar{R}^5, \bar{R}^6)$$

and thus $w\big(f(\bar{\gamma})\big) = 1$. Let $\boldsymbol{R} \in \mathcal{L}^n$. By Lemmas 4 (b), 5 (b), and 8,

$$w\big(f(\gamma_1^{\boldsymbol{R}})\big) + \dots + w\big(f(\gamma_n^{\boldsymbol{R}})\big) = 1. \qquad (2)$$

Since $\text{length}(f(\gamma_i^{\boldsymbol{R}})) = 6$ for all $i \in N$, $|w(f(\gamma_i^{\boldsymbol{R}}))| \leq 1$ for all $i \in N$. By Lemma 5 (a), there exists $i \in N$ such that

$$w\big(f(\gamma_i^{\boldsymbol{R}})\big) = 1. \tag{3}$$

Fix such an agent i. Let $\boldsymbol{R}' \in \mathcal{L}^n$. By Lemma 7, $\gamma_i^{\boldsymbol{R}} \simeq \gamma_i^{\boldsymbol{R}'}$. Then Lemma 4 (b) implies $f(\gamma_i^{\boldsymbol{R}}) \simeq f(\gamma_i^{\boldsymbol{R}'})$, and thus $w\big(f(\gamma_i^{\boldsymbol{R}})\big) = w\big(f(\gamma_i^{\boldsymbol{R}'})\big)$ by Lemma 5 (b). By equality (3), $w\big(f(\gamma_i^{\boldsymbol{R}'})\big) = 1$, and $f(\gamma_i^{\boldsymbol{R}'}) = \bar{c}_{f(\boldsymbol{R}')}$ by Lemma 5 (b) and Lemma 6. Since $\boldsymbol{R}'$ is taken arbitrarily, agent i is a manipulator.

Let i be the manipulator. Let $j \neq i$ and $\boldsymbol{R} \in \mathcal{L}^n$. We can show that $w\big(f(\gamma_j^{\boldsymbol{R}})\big) = 0$. Suppose not. Then by equality (2), there exists $j' \neq i$ such that $w\big(f(\gamma_{j'}^{\boldsymbol{R}})\big) = 1$. The same argument as above shows that for each $\boldsymbol{R}' \in \mathcal{L}^n$, $f(\gamma_{j'}^{\boldsymbol{R}'}) = \bar{c}_{f(\boldsymbol{R}')}$. Let $\bar{\boldsymbol{R}} = (\bar{R}^l, \bar{R}^l, \ldots, \bar{R}^l) \in \mathcal{L}^n$. By *adjacency-extended unanimity*, $f(\bar{\boldsymbol{R}}) = \bar{R}^l$, and thus $f(\gamma_{j'}^{\bar{\boldsymbol{R}}}) = \bar{c}_{\bar{R}^l}$. Since this implies $f(\bar{R}^{l+1}, \bar{\boldsymbol{R}}_{-j'}) = \bar{R}^{l+1}$, $f(\gamma_i^{(\bar{R}^{l+1}, \bar{\boldsymbol{R}}_{-j'})}) = \bar{c}_{\bar{R}^{l+1}}$, and thus $f(\bar{R}^{l+1}, \bar{R}^{l+1}, \bar{\boldsymbol{R}}_{-(i,j')}) = \bar{R}^{l+2}$. However, this contradicts *adjacency-extended unanimity*. Therefore $w\big(f(\gamma_j^{\boldsymbol{R}})\big) = 0$. Since each loop $\hat{c} \in C$ such that $w(\hat{c}) = 0$ and $\text{length}(\hat{c}) > 1$ must include two integers k, k' ($1 \leq k < k' \leq \text{length}(\hat{c})$) such that $\hat{c}(k) = \hat{c}(k')$, and $\gamma_j^{\boldsymbol{R}}$ consists of six preference profiles, the set $\{f(\gamma_j^{\boldsymbol{R}})(1), \ldots, f(\gamma_j^{\boldsymbol{R}})(6)\}$ contains less than six preferences. This implies that j ($\neq i$) cannot be a manipulator.

Hence we showed the following result:

Proposition 9. *Suppose $m = 3$. If a social welfare function f satisfies* bounded response *and* adjacency-extended unanimity, *then there exists a unique manipulator $i \in N$.*

We note that the existence of a manipulator rules out most, if not all, plausible social welfare functions. One may wonder if the manipulator in Proposition 9 is a dictator in each social welfare function satisfying *bounded response* and *adjacency-extended unanimity*. We have the following counterexample: Recall that *Pareto efficiency* is stronger than *adjacency-extended unanimity*.

Example 1. Let $n = 3$. Then the following social welfare function f satisfies *bounded response* and *Pareto efficiency*, but is not dictatorial. Let us denote $R_1 = \bar{R}^l$.

$$f(\bar{R}^l, R_2, R_3) = \begin{cases} \bar{R}^{l+1} & \text{if } (R_2, R_3) = (\bar{R}^1, \bar{R}^4), \\ \bar{R}^l & \text{otherwise.} \end{cases}$$

In this case, agent 1 is the manipulator, but if $(R_2, R_3) = (\bar{R}_1, \bar{R}_4)$, the social preference is not the same as R_1. *Pareto efficiency* does not help because $\bar{R}_4$ is the inverse of $\bar{R}_1$.

Example 1 shows that a manipulator may not be a dictator. We can, however, show the following result that dictatorship prevails for many preference profiles.

Proposition 10. *Under the assumptions in Proposition 9, let $i \in N$ be the manipulator. For each $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$, if there exist two adjacent preferences $R, R' \in \mathcal{L}$ such that $R_j \in \{R, R'\}$ for each $j \in N \setminus \{i\}$, then $f(R_i, \boldsymbol{R}_{-i}) = R_i$ for each $R_i \in \mathcal{L}$.*

Proof. We can assume without loss of generality that $R = \bar{R}^1$ and $R' = \bar{R}^2$. Fix $\boldsymbol{R}_{-i} \in \{\bar{R}^1, \bar{R}^2\}^{n-1}$ arbitrarily. By *adjacency-extended unanimity*, $\big(f(\bar{R}^1, \boldsymbol{R}_{-i}), f(\bar{R}^2, \boldsymbol{R}_{-i})\big) = (\bar{R}^1, \bar{R}^1)$, $(\bar{R}^1, \bar{R}^2)$, $(\bar{R}^2, \bar{R}^1)$, or $(\bar{R}^2, \bar{R}^2)$. Let $R_i = \bar{R}^1$. Among these four possibilities, only $(\bar{R}^1, \bar{R}^2)$ satisfies equality (3). Therefore, $f(\gamma_i^{\boldsymbol{R}}, \boldsymbol{R}_{-i}) = \bar{c}_{\bar{R}^1}$. This implies the proposition. □

In the case of two agents, the supposition of Proposition 10 is always met, and f is dictatorial.

If we assume *Pareto efficiency*, dictatorship prevails for even more preference profiles.

Proposition 11. *Under the assumptions in Proposition 9, let $i \in N$ be the manipulator. Suppose that f satisfies* Pareto efficiency. *Then for each $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$, if there exist two alternatives $x, y \in X$ such that $x \; R_j \; y$ for each $j \in N \setminus \{i\}$, then $f(R_i, \boldsymbol{R}_{-i}) = R_i$ for each $R_i \in \mathcal{L}$.*

Proof. We can assume without loss of generality that for each $j \in N \setminus \{i\}$, $x^1 \; R_j \; x^2$, or equivalently, $R_j = \bar{R}^1$, $\bar{R}^2$, or $\bar{R}^3$. Let $f(\bar{R}^1, \boldsymbol{R}_{-i}) = \bar{R}^l$ $(1 \leq l \leq 6)$. Then by (3), we have $f(\gamma_i^{(\bar{R}^1, \boldsymbol{R}_{-i})}) = \bar{c}_{\bar{R}^l}$. Since $f(\bar{R}^{l'}, \boldsymbol{R}_{-i}) = \bar{R}^{l+l'-1}$, and *Pareto efficiency* implies $f(\bar{R}^{l'}, \boldsymbol{R}_{-i}) = \bar{R}^1$, $\bar{R}^2$, or $\bar{R}^3$ for each $l' = 1, 2, 3$, we have $l = 1$. Again by $f(\gamma_i^{(\bar{R}^1, \boldsymbol{R}_{-i})}) = \bar{c}_{\bar{R}^1}$, we proved that $f(R_i, \boldsymbol{R}_{-i}) = R_i$ for all $R_i \in \mathcal{L}$. □

4 Four or more alternatives

Since *bounded response* deviates from the independence property, the result in the three-alternative case is not immediately generalized to the four-or-more-alternative case. In this section, we show that *bounded response* and *adjacency-extended unanimity* imply dictatorship when there are $m \geq 4$ alternatives in the society. Thus, we have no social welfare function like that in Example 1 in the three-alternative case.

Let us fix a preference $\bar{R} \in \mathcal{L}$ arbitrarily. Let $x^k = r^k(\bar{R}) \in X$ for each $k = 1, \ldots, m$.

For each t $(1 \leq t \leq m-2)$, we introduce a subset $\mathcal{D}_t$ of $\mathcal{L}$, consisting of preferences $R \in \mathcal{L}$ such that $r^k(R) = x^k$ unless $k = t$, $t+1$, or $t+2$. For each t $(1 \leq t \leq m-2)$, the domain $\mathcal{D}_t$ comprises six preferences $\bar{R}^{1,t}, \ldots, \bar{R}^{6,t}$ presented in Table 2, in which $\bar{R}^{1,t} = \bar{R}$ for each t. By rule, for each integer l', let $\bar{R}^{l',t} = \bar{R}^{l,t}$ if $l' = 6p + l$ for some integer p.

$\bar{R}^{1,t}$	$\bar{R}^{2,t}$	$\bar{R}^{3,t}$	$\bar{R}^{4,t}$	$\bar{R}^{5,t}$	$\bar{R}^{6,t}$
x^1	x^1	x^1	x^1	x^1	x^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}
x^t	x^t	x^{t+2}	x^{t+2}	x^{t+1}	x^{t+1}
x^{t+1}	x^{t+2}	x^t	x^{t+1}	x^{t+2}	x^t
x^{t+2}	x^{t+1}	x^{t+1}	x^t	x^t	x^{t+2}
x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^m	x^m	x^m	x^m	x^m	x^m

Table 2: Preferences in $\mathcal{D}_t$. We note that $\bar{R}^{1,t} = \bar{R}$ for each t ($1 \leq t \leq m-2$).

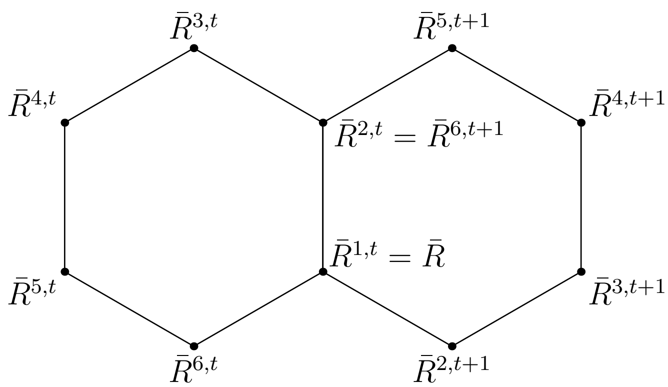

Figure 2: A graph representing the adjacent relations in $\mathcal{D}_t \cup \mathcal{D}_{t+1}$.

In this notation, $\bar{R}^{l,t}$ and $\bar{R}^{l',t}$ are adjacent if and only if there exists an integer p such that $l - l' = 6p + 1$ or $6p - 1$. We note that for each t ($1 \leq t \leq m-3$), $\bar{R}^{2,t} = \bar{R}^{6,t+1}$ and $\mathcal{D}_t \cap \mathcal{D}_{t+1} = \{\bar{R}, \bar{R}^{2,t}\}$. Thus for each t ($1 \leq t \leq m-3$), $\mathcal{D}_t \cup \mathcal{D}_{t+1}$ comprises ten preferences presented in the graph in Figure 2, in which each pair of adjacent preferences is connected with an edge.

For each t ($1 \leq t \leq m-2$) and each l ($1 \leq l \leq 6$), let $\bar{c}_{\bar{R}^{l,t},t} = (\bar{R}^{l,t}, \bar{R}^{l+1,t}, \ldots, \bar{R}^{l+5,t}) \in C$. We can show the following lemma as in the proof of Lemma 6.

Lemma 12. *Suppose that $c \in C$ is a loop of length 6.*

(a) If $c(1) = \bar{R}$, then either $c \simeq (\bar{R}) \in C$, or there exists t ($1 \leq t \leq m-2$) such that $c = \bar{c}_{\bar{R},t}$ or $(\bar{c}_{\bar{R},t})^{-1}$.

(b) If there exists t ($1 \leq t \leq m-2$) such that $c(1) \in \mathcal{D}_t$ and $c \simeq \bar{c}_{\bar{R},t}$, then $c = \bar{c}_{c(1),t}$.

The proof is given in Appendix A.3.

For each $i \in N$, each t $(1 \leq t \leq m-2)$, and each $\boldsymbol{R} \in \mathcal{L}^n$ such that $R_i \in \mathcal{D}_t$, let $\gamma_i^{\boldsymbol{R},t} \in \Gamma$ be such that length$(\gamma_i^{\boldsymbol{R},t}) = 6$, and for each k $(1 \leq k \leq 6)$, $(\gamma_i^{\boldsymbol{R},t}(k))_i = \bar{c}_{R_i,t}(k)$, and $(\gamma_i^{\boldsymbol{R},t}(k))_j = R_j$ for all $j \neq i$.

Let $\bar{\boldsymbol{R}} = (\bar{R}, \bar{R}, \ldots, \bar{R}) \in \mathcal{L}^n$.

Lemma 13. *Suppose that f satisfies* bounded response *and* adjacency-extended unanimity. *Then there exists a unique agent $i^* \in N$ such that $f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t}) = \bar{c}_{\bar{R},t}$ for each t $(1 \leq t \leq m-2)$.*

The proof is provided in Appendix A.4.

Let $i^* \in N$ be the agent in Lemma 13, and fix $\boldsymbol{R}_{-i^*} \in \mathcal{L}^{n-1}$ arbitrarily. Since $\gamma_{i^*}^{\bar{\boldsymbol{R}},t} \simeq \gamma_{i^*}^{(\bar{R},\boldsymbol{R}_{-i^*}),t}$ for each t $(1 \leq t \leq m-2)$ by similar arguments in Lemma 7, we have $f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t}) \simeq f(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}_{-i^*}),t})$ for each t $(1 \leq t \leq m-2)$ by Lemma 4. We can show the following:

Lemma 14. *Suppose that f satisfies* bounded response *and* adjacency-extended unanimity. *Then for each $\boldsymbol{R}_{-i^*} \in \mathcal{L}^{n-1}$, $f(\bar{R}, \boldsymbol{R}_{-i^*}) = \bar{R}$.*

Proof. We assume that $f(\bar{R}, \boldsymbol{R}_{-i^*}) \neq \bar{R}$ and derive a contradiction. Consider a sequence of profiles $\boldsymbol{R}^1, \boldsymbol{R}^2, \ldots, \boldsymbol{R}^l$ such that $\boldsymbol{R}^1 = \bar{\boldsymbol{R}}$, $\boldsymbol{R}^l = (\bar{R}, \boldsymbol{R}_{-i^*})$, $R^k_{i^*} = \bar{R}$, and $\boldsymbol{R}^k$ and $\boldsymbol{R}^{k+1}$ are adjacent for each k $(1 \leq k \leq l-1)$. Since f satisfies *bounded response*, $f(\boldsymbol{R}^k)$ and $f(\boldsymbol{R}^{k+1})$ are either adjacent or the same for each k $(1 \leq k \leq l-1)$. Since $f(\boldsymbol{R}^1) = \bar{R}$ and $f(\boldsymbol{R}^l) \neq \bar{R}$, there is k' $(2 \leq k' \leq l)$ such that $f(\boldsymbol{R}^{k'}) = R'$ is adjacent to $\bar{R}$. Let $(\bar{R}, \boldsymbol{R}'_{-i^*}) = \boldsymbol{R}^{k'}$. Since R' is adjacent to $\bar{R}$, R' is obtained by exchanging the positions of one pair of consecutively ranked alternatives in $\bar{R}$. Thus, by the definition of domains $\mathcal{D}_t$, (i) $R' = \bar{R}^{6,t}$ for some t $(1 \leq t \leq m-3)$, or (ii) $R' = \bar{R}^{2,t}$ for some t $(2 \leq t \leq m-2)$ (or both).

Note that $\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t} \simeq \gamma_{i^*}^{\bar{\boldsymbol{R}},t}$ by Lemma 7. By Lemma 4, $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big) \simeq f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t})$. By Lemma 13, $f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t}) = \bar{c}_{\bar{R},t}$. Thus, $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big) \simeq \bar{c}_{\bar{R},t}$.

Before going into further arguments, we explain a rough idea of the proof. Suppose that $f(\bar{R}, \boldsymbol{R}_{-i^*}) = \bar{R}^{6,t}$ for some t $(1 \leq t \leq m-3)$. We consider the changes of agent i^*'s preferences (1) $\bar{R}(= \bar{R}^{1,t}) \to \bar{R}^{2,t}$ $(= \bar{R}^{6,t+1})$, and then (2) $\bar{R}^{6,t+1} \to \bar{R}$ $(= \bar{R}^{1,t+1})$ (dashed arrows in Figure 3). Along with the change of agent i^*'s preferences, the social preference changes. Since the agent i^*'s preference returns to the initial preference $\bar{R}$, the social preference should also return to the initial one. However, we will show that the social preference changes as the solid arrows in Figure 3, which is a contradiction. A similar argument follows when $f(\bar{R}, \boldsymbol{R}_{-i^*})$ is another preference. Now, we start formal arguments by considering the above two cases in order:

Case (i): $R' = \bar{R}^{6,t}$ for some t $(1 \leq t \leq m-3)$. Since $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big)(1) = \bar{R}^{6,t} \in \mathcal{D}_t$ and $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big) \simeq \bar{c}_{\bar{R},t}$, we have $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big) = c_{\bar{R}^{6,t},t}$ by Lemma 12 (b). Therefore,

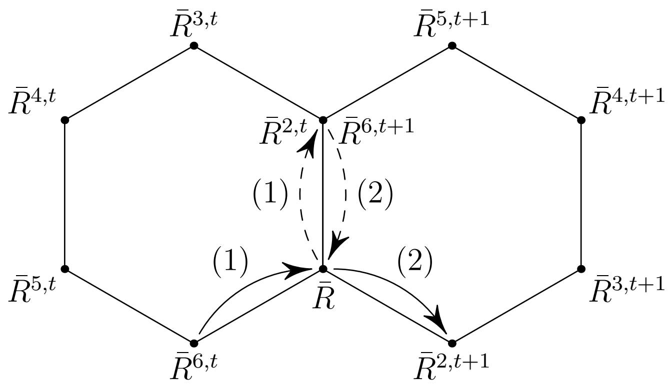

Figure 3: The changes of an agent's preferences (dashed arrows), and the changes of social preferences (solid arrows).

$f(\bar{R}^{2,t}, \boldsymbol{R}'_{-i^*}) = f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big)(2) = c_{\bar{R}^{6,t},t}(2) = \bar{R}$. Since $\bar{R}^{2,t} = \bar{R}^{6,t+1}$, $f(\bar{R}^{6,t+1}, \boldsymbol{R}'_{-i^*}) = \bar{R}$. By Lemma 7, $\gamma_{i^*}^{(\bar{R}^{6,t+1},\boldsymbol{R}'_{-i^*}),t+1} \simeq \gamma_{i^*}^{\bar{\boldsymbol{R}},t+1}$. By Lemma 4, $f(\gamma_{i^*}^{(\bar{R}^{6,t+1},\boldsymbol{R}'_{-i^*}),t+1}) \simeq f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t+1})$. Since $f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t+1}) = \bar{c}_{\bar{R},t+1}$ by Lemma 13, we have $f(\gamma_{i^*}^{(\bar{R}^{6,t+1},\boldsymbol{R}'_{-i^*}),t+1}) \simeq \bar{c}_{\bar{R},t+1}$. Since $f(\gamma_{i^*}^{(\bar{R}^{6,t+1},\boldsymbol{R}'_{-i^*}),t+1})(1) = \bar{R} \in \mathcal{D}_{t+1}$ and $f(\gamma_{i^*}^{(\bar{R}^{6,t+1},\boldsymbol{R}'_{-i^*}),t+1}) \simeq \bar{c}_{\bar{R},t+1}$, we have $f\big(\gamma_{i^*}^{(\bar{R}^{6,t+1},\boldsymbol{R}'_{-i^*}),t+1}\big) = \bar{c}_{\bar{R},t+1}$ by Lemma 12 (b). This implies $R' = f(\bar{R}, \boldsymbol{R}'_{-i^*}) = f\big(\gamma_{i^*}^{(\bar{R}^{6,t+1},\boldsymbol{R}'_{-i^*}),t+1}\big)(2) = \bar{c}_{\bar{R},t+1}(2) = \bar{R}^{2,t+1}$, which contradicts the assumption of $R' = \bar{R}^{6,t}$.

Case (ii): $R' = \bar{R}^{2,t}$ for some t ($2 \leq t \leq m-2$). Since $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big)(1) = \bar{R}^{2,t} \in \mathcal{D}_t$ and $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big) \simeq \bar{c}_{\bar{R},t}$, we have $f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big) = c_{\bar{R}^{2,t},t}$ by Lemma 12 (b). Therefore, $f(\bar{R}^{6,t}, \boldsymbol{R}'_{-i^*}) = f\big(\gamma_{i^*}^{(\bar{R},\boldsymbol{R}'_{-i^*}),t}\big)(6) = c_{\bar{R}^{2,t},t}(6) = \bar{R}$. Since $\bar{R}^{6,t} = \bar{R}^{2,t-1}$, $f(\bar{R}^{2,t-1}, \boldsymbol{R}'_{-i^*}) = \bar{R}$. By Lemma 7, $\gamma_{i^*}^{(\bar{R}^{2,t-1},\boldsymbol{R}'_{-i^*}),t-1} \simeq \gamma_{i^*}^{\bar{\boldsymbol{R}},t-1}$. By Lemma 4, $f(\gamma_{i^*}^{(\bar{R}^{2,t-1},\boldsymbol{R}'_{-i^*}),t-1}) \simeq f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t-1})$. Since $f(\gamma_{i^*}^{\bar{\boldsymbol{R}},t-1}) = \bar{c}_{\bar{R},t-1}$ by Lemma 13, we have $f(\gamma_{i^*}^{(\bar{R}^{2,t-1},\boldsymbol{R}'_{-i^*}),t-1}) \simeq \bar{c}_{\bar{R},t-1}$. Since $f\big(\gamma_{i^*}^{(\bar{R}^{2,t-1},\boldsymbol{R}'_{-i^*}),t-1}\big)(1) = \bar{R} \in \mathcal{D}_{t-1}$ and $f(\gamma_{i^*}^{(\bar{R}^{2,t-1},\boldsymbol{R}'_{-i^*}),t-1}) \simeq \bar{c}_{\bar{R},t-1}$, we have $f\big(\gamma_{i^*}^{(\bar{R}^{2,t-1},\boldsymbol{R}'_{-i^*}),t-1}\big) = \bar{c}_{\bar{R},t-1}$ by Lemma 12 (b). This implies $R' = f(\bar{R}, \boldsymbol{R}'_{-i^*}) = f\big(\gamma_{i^*}^{(\bar{R}^{2,t-1},\boldsymbol{R}'_{-i^*}),t-1}\big)(6) = \bar{c}_{\bar{R},t-1}(6) = \bar{R}^{6,t-1}$, which contradicts the assumption of $R' = \bar{R}^{2,t}$. □

We have shown that for the fixed $\bar{R} \in \mathcal{L}$, there is $i^* \in N$ such that $f(\bar{R}, \boldsymbol{R}_{-i^*}) = \bar{R}$ for each $\boldsymbol{R}_{-i^*}$. Since $\bar{R}$ was arbitrary, we have shown that for each $R \in \mathcal{L}$, there is $i^*(R) \in N$ such that $f(R, \boldsymbol{R}_{-i^*(R)}) = R$ for each $\boldsymbol{R}_{-i^*(R)}$. Also, it can be seen that for each $R, R' \in \mathcal{L}$, $i^*(R) = i^*(R')$. (Suppose $i^*(R) \neq i^*(R')$. Then, for $\boldsymbol{R} \in \mathcal{L}$ such that $R_{i^*(R)} = R$ and $R_{i^*(R')} = R'$, we have $f(\boldsymbol{R}) = R$ and $f(\boldsymbol{R}) = R'$, which is impossible.)

Thus we showed the following result:

Theorem 1. *Suppose $m \geq 4$. If a social welfare function f satisfies* bounded response *and* adjacency-extended unanimity, *then f is dictatorial.*

5 Weak orders

In the main sections, we discussed *bounded response* in the set of linear orders $\mathcal{L}$, which rules out preferences including ties. In this section, we generalize our impossibility in Theorem 1 when ties are allowed in social preferences.[5]

Let $\mathcal{R}$ be the set of all preference relations on X which are complete and transitive. We define adjacency between two preferences $R_i, R_i' \in \mathcal{R}$ the same as in $\mathcal{L}$: Preferences R_i and R_i' are *adjacent* if $R_i \neq R_i'$, and there exist two alternatives $x, y \in X$ such that for each pair of alternatives $z, w \in X$ with $\{z, w\} \neq \{x, y\}$, $z \; R_i \; w$ if and only if $z \; R_i' \; w$. *Bounded response* is defined the same as Axiom 1: A social welfare function $f : \mathcal{L}^n \to \mathcal{R}$ satisfies *bounded response* if adjacency between two linear orders $\boldsymbol{R}, \boldsymbol{R}' \in \mathcal{L}^n$ implies that $f(\boldsymbol{R})$ and $f(\boldsymbol{R}')$ are adjacent or the same. We note that *bounded response* is weaker than independence of irrelevant alternatives in this environment. *Adjacency-extended unanimity* is also defined the same as Axiom 2: A social welfare function $f : \mathcal{L}^n \to \mathcal{R}$ satisfies *adjacency-extended unanimity* if for each $\boldsymbol{R} \in \mathcal{L}^n$, if $\{R_i \mid i \in N\} = \{R\}$ then $f(\boldsymbol{R}) = R$, and if $\{R_i \mid i \in N\} = \{R, R'\}$ for some adjacent $R, R' \in \mathcal{L}$, then $f(\boldsymbol{R})$ is R, R', or the weak order adjacent to both R and R'.

We say that, given a social welfare function $f : \mathcal{L}^n \to \mathcal{R}$, an agent $i \in N$ is a *dictator* if for each $R_i \in \mathcal{L}$ and each $\boldsymbol{R}_{-i} \in \mathcal{L}^{n-1}$, $f(R_i, \boldsymbol{R}_{-i}) = R_i$. The social welfare function f is *dictatorial* if there is a dictator.

Proposition 15. *Suppose $m \geq 4$. If a social welfare function f satisfies* bounded response *and* adjacency-extended unanimity, *then f is dictatorial.*

The proof is provided in Appendix A.5.

6 Conclusion

We have shown an impossibility result, which is stronger than Arrow's impossibility result, in the case with four or more alternatives. A pair of linear orders is called adjacent if

[5]We do not consider ties in agents' preferences. Considering ties in agents' preferences involves technical complexity which is beyond the scope of our interests. (Our conjecture is that by appropriately modifying the definition of bounded response, our impossibility holds.) Also, for each social welfare function defined over weak orders, by restricting its domain to linear orders, our impossibility result applies. Thus, we can never escape from a flavor of impossibility by considering weak orders as agents' preferences.

they disagree only between two alternatives. *Bounded response* requires that if an agent's preferences are adjacent and the others' preferences are the same, then the social preferences should be adjacent or the same. *Adjacency-extended unanimity* requires unanimity, and that if each agent's preference is either of two adjacent preferences, then the social preference must be one of the two. When the society has four or more alternatives, we showed that if a social welfare function satisfies these two axioms, then it must be dictatorial. Since *bounded response* is weaker than *independence of irrelevant alternatives*, this impossibility result is stronger than Arrow's.

Despite the existence of a counterexample in the three-alternative case, we showed that the set of non-dictatorial social welfare functions satisfying both *bounded response* and *adjacency-extended unanimity* is small. Specifically, if there are three alternatives, there exists a unique agent that can manipulate the social preference as he likes if he knows the others' preferences. This requirement rules out most, if not all, plausible social welfare functions.

Although the above arguments assume that the preferences are linear orders, we also show that our impossibility result carries over to the case with social preferences with ties. This extension is quite straightforward.

A Appendix: Proofs of the Results

A.1 Proof of Lemma 5

We first show the "only-if part" of (b), and then prove the entire proposition.

Lemma 16. *For each pair of loops $c, c' \in C$, $c \simeq c'$ implies $w(c) = w(c')$.*

Proof. It suffices to prove $w(c) = w(c')$ whenever c and c' satisfy either of conditions (1), (2), or (3) in Definition 1.

First suppose that c and c' satisfy condition (1) in Definition 1. Then $\text{length}(c') = \text{length}(c) + 1$, and there is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(c)$) such that $c'(k) = c(k)$ if $1 \leq k \leq \bar{k}$, and $c'(k) = c(k-1)$ if $\bar{k} + 1 \leq k \leq \text{length}(c')$. This implies that

$$\alpha\big(c'(k), c'(k+1)\big) = \begin{cases} \alpha\big(c(k), c(k+1)\big) & \text{if } 1 \leq k \leq \bar{k} - 1, \\ \alpha\big(c(\bar{k}), c(\bar{k})\big) = 0 & \text{if } k = \bar{k}, \\ \alpha\big(c(k-1), c(k)\big) & \text{if } \bar{k} + 1 \leq k \leq \text{length}(c), \end{cases}$$

and thus $w(c) = w(c')$.

Second if c and c' satisfy condition (2) in Definition 1, we can prove $w(c) = w(c')$ in a similar way as above.

Third suppose that c and c' satisfy condition (3) in Definition 1. Then $\text{length}(c') = \text{length}(c)$. If $\text{length}(c) = \text{length}(c') = 1$, $w(c) = w(c') = 0$ is obvious. We assume $\text{length}(c) = \text{length}(c') > 1$. There is $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(c)$) such that $c'(k) = c(k)$ whenever $k \neq \bar{k}$ ($1 \leq k \leq \text{length}(c)$). Since $\alpha\big(c'(k), c'(k+1)\big) = \alpha\big(c(k), c(k+1)\big)$ whenever $k \neq \bar{k} - 1, \bar{k}$, we have

$$\begin{aligned} &w(c) - w(c') \\ &\quad = \alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) - \alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) - \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big). \end{aligned}$$

Let $c(\bar{k}) = \bar{R}^l$. Since c is a loop, $c(\bar{k}-1)$ equals either $\bar{R}^{l-1}$, $\bar{R}^l$, or $\bar{R}^{l+1}$, and also $c(\bar{k}+1)$ equals $\bar{R}^{l-1}$, $\bar{R}^l$, or $\bar{R}^{l+1}$. Thus, there are $3 \times 3 = 9$ cases as follows:

Case 1: $c(\bar{k}-1) = \bar{R}^{l-1}$ and $c(\bar{k}+1) = \bar{R}^{l-1}$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = 1/6 - 1/6 = 0$. Since $c'(\bar{k})$ is adjacent or equal to both $c(\bar{k}-1)$ and $c(\bar{k}+1)$, $c'(\bar{k})$ must be either $\bar{R}^{l-2}$, $\bar{R}^{l-1}$, or $\bar{R}^l$. In any case, $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = 0$.

Case 2: $c(\bar{k}-1) = \bar{R}^{l-1}$ and $c(\bar{k}+1) = \bar{R}^l$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = 1/6 + 0 = 1/6$. Then $c'(\bar{k})$ must be either $\bar{R}^{l-1}$ or $\bar{R}^l$. In any case, $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = 1/6$.

Case 3: $c(\bar{k}-1) = \bar{R}^{l-1}$ and $c(\bar{k}+1) = \bar{R}^{l+1}$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = 1/6 + 1/6 = 2/6$. Then $c'(\bar{k})$ must be $\bar{R}^l$, and we have $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = 2/6$.

Case 4: $c(\bar{k}-1) = \bar{R}^l$ and $c(\bar{k}+1) = \bar{R}^{l-1}$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = 0 - 1/6 = -1/6$. Then $c'(\bar{k})$ must be either $\bar{R}^{l-1}$ or $\bar{R}^l$. In any case, $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = -1/6$.

Case 5: $c(\bar{k}-1) = \bar{R}^l$ and $c(\bar{k}+1) = \bar{R}^l$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = 0 + 0 = 0$. Then $c'(\bar{k})$ must be either $\bar{R}^{l-1}$, $\bar{R}^l$, or $\bar{R}^{l+1}$. In any case, $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = 0$.

Case 6: $c(\bar{k}-1) = \bar{R}^l$ and $c(\bar{k}+1) = \bar{R}^{l+1}$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = 0 + 1/6 = 1/6$. Then $c'(\bar{k})$ must be either $\bar{R}^l$ or $\bar{R}^{l+1}$. In any case, $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = 1/6$.

Case 7: $c(\bar{k}-1) = \bar{R}^{l+1}$ and $c(\bar{k}+1) = \bar{R}^{l-1}$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = -1/6 - 1/6 = -2/6$. Then $c'(\bar{k})$ must be $\bar{R}^l$, and we have $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = -2/6$.

Case 8: $c(\bar{k}-1) = \bar{R}^{l+1}$ and $c(\bar{k}+1) = \bar{R}^{l}$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = -1/6 + 0 = -1/6$. Then $c'(\bar{k})$ must be either $\bar{R}^l$ or $\bar{R}^{l+1}$. In any case, $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = -1/6$.

Case 9: $c(\bar{k}-1) = \bar{R}^{l+1}$ and $c(\bar{k}+1) = \bar{R}^{l+1}$, thereby $\alpha\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha\big(c(\bar{k}), c(\bar{k}+1)\big) = -1/6 + 1/6 = 0$. Then $c'(\bar{k})$ must be either $\bar{R}^l$, $\bar{R}^{l+1}$, or $\bar{R}^{l+2}$. In any case, $\alpha\big(c'(\bar{k}-1), c'(\bar{k})\big) + \alpha\big(c'(\bar{k}), c'(\bar{k}+1)\big) = 0$.

Therefore $w(c) = w(c')$ in all cases. □

Proof of Lemma 5. Let $c \in C$. Let $\hat{c} \in C$ be the shortest loop among the loops connected to c, i.e., $\hat{c} \simeq c$ and there is no $c'' \in C$ such that $c'' \simeq c$ and $\text{length}(c'') < \text{length}(\hat{c})$. If $\text{length}(\hat{c}) = 1$, then $\hat{c} \simeq (c(1))$, and hence $c \simeq (c(1))$. By Lemma 16, $w(c) = w((c(1))) = 0$.

Assume that $\text{length}(\hat{c}) \geq 2$. Then for each k ($1 \leq k \leq \text{length}(\hat{c})$), we have (i) $\alpha\big(\hat{c}(k), \hat{c}(k+1)\big) \neq 0$, and (ii) $\alpha\big(\hat{c}(k), \hat{c}(k+1)\big) + \alpha\big(\hat{c}(k+1), \hat{c}(k+2)\big) \neq 0$. This is because of the following: If (i) fails, then $\hat{c} \simeq \big(\hat{c}(1), \dots, \hat{c}(k), \hat{c}(k+2), \dots, \hat{c}(\text{length}(\hat{c}))\big)$ by (2) in Definition 1. If (ii) fails, then $\hat{c}(k) = \hat{c}(k+2)$ and thus $\hat{c} \simeq \big(\hat{c}(1), \dots, \hat{c}(k), \hat{c}(k+3), \dots, \hat{c}(\text{length}(\hat{c}))\big)$ by (3) and (2) in Definition 1. In both cases, there exists c'' such that $c'' \simeq \hat{c}$ and $\text{length}(c'') < \text{length}(\hat{c})$.

Therefore we must have either $\alpha\big(\hat{c}(k), \hat{c}(k+1)\big) = 1$ for all k, or $\alpha\big(\hat{c}(k), \hat{c}(k+1)\big) = -1$ for all k. This implies that there exist an integer p and $R \in \mathcal{L}$ such that $\hat{c} = \bar{c}_R^p$. Since Lemma 2 shows that $\bar{c}_R \simeq \bar{c}_{\bar{R}^1}$ for each $R \in \mathcal{L}$, we have $c \simeq \bar{c}_{\bar{R}^1}^p$. In this case, $w(c) = w(\bar{c}_{\bar{R}^1}^p) = p$ by Lemma 16, and the definition of w.

In any case, $w(c)$ is an integer, and $\bar{c}_{\bar{R}^1}^{w(c)}$ is the shortest in the set of connected loops to c. Hence (a), (c), and (d) are proved. Since we showed that $c \simeq \bar{c}_{\bar{R}^1}^{w(c)}$ for all $c \in C$, if $w(c) = w(c')$, then $c \simeq \bar{c}_{\bar{R}^1}^{w(c)} \simeq c'$. Combined with Lemma 16, we showed (b). □

A.2 Proof of Lemma 8

We first show a general lemma: For each loop of preference profiles $\gamma \in \Gamma$ and each agent $i \in N$, we can show that γ is connected to a loop of preference profiles given by replacing agent i's preferences in γ by $(\gamma(1))_i$.

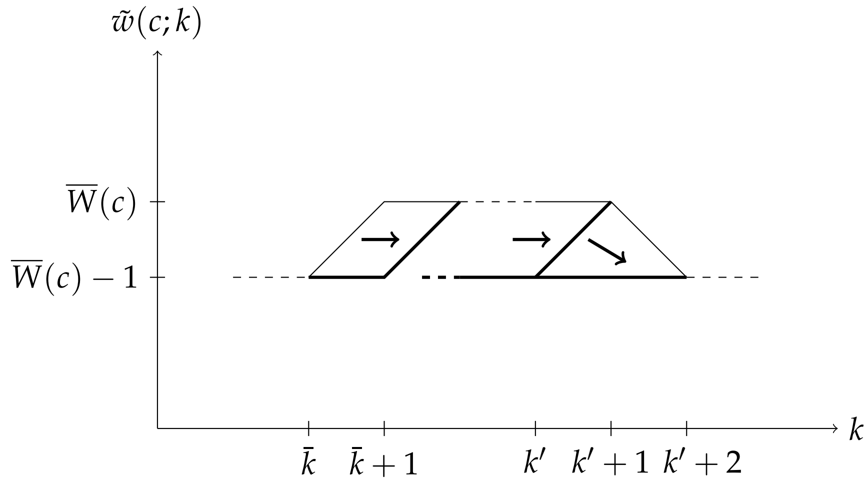

Figure 4: Deformation of a path in the proof of Lemma 17

Lemma 17. *Let* $\gamma \in \Gamma$, $i \in N$, $K = \text{length}(\gamma)$, *and* $c = \big((\gamma(1))_i, (\gamma(2))_i, \ldots, (\gamma(K))_i\big)$. *If* $w(c) = 0$, *then* $\gamma \simeq \hat{\gamma}$ *where for each* k $(1 \leq k \leq K)$,

$$(\hat{\gamma}(k))_j = \begin{cases} c(1) & \text{if } j = i, \\ (\gamma(k))_j & \text{if } j \neq i. \end{cases}$$

Moreover, in the process of deformation from γ *to* $\hat{\gamma}$, *the first profile remains the same.*

Proof. For each $c' \in C$ and each k $(1 \leq k \leq \text{length}(c'))$, let $\tilde{w}(c';k) = \sum_{k'=1}^{k} \alpha\big(c'(k'), c'(k'+1)\big)$, $\overline{W}(c') = \max_{1 \leq k' \leq \text{length}(c')} \tilde{w}(c';k')$, $\underline{W}(c') = \min_{1 \leq k' \leq \text{length}(c')} \tilde{w}(c';k')$, and $\overline{\mathcal{K}}(c') = \{k' \mid \tilde{w}(c';k') = \overline{W}(c)\}$. By the definition of $\overline{\mathcal{K}}$, $\overline{\mathcal{K}}(c) \neq \emptyset$.

We show that if $\overline{W}(c) \geq 1$, then there exist $\gamma^* \in \Gamma$ and the corresponding loop c^* of agent i's preferences such that $\gamma^* \simeq \gamma$, $w(c^*) = 0$, $\overline{W}(c^*) = \overline{W}(c) - 1$, and $\underline{W}(c^*) = \underline{W}(c)$. Assume $\overline{W}(c) \geq 1$. Let $\bar{k}, k' \in \overline{\mathcal{K}}(c)$ be such that $\bar{k} \leq k'$, $\alpha(c(\bar{k}), c(\bar{k}+1)) = 1$, $\alpha(c(k), c(k+1)) = 0$ for each k $(\bar{k}+1 \leq k \leq k')$, and $\alpha(c(k'+1), c(k'+2)) = -1$. Since $w(c) = 0$, such $\bar{k}$ and k' exist. (It is possible that $\bar{k} = k'$.) When we draw the graph of $\tilde{w}(c;\cdot)$, the graph corresponding to the interval from $\bar{k}+1$ to $k'+1$ forms a (flat) peak at $\overline{W}(c)$. (Thin lines in Figure 4.) In the following, we scrape the peak from the left, and make the graph flat at $\overline{W}(c) - 1$.

In γ, replace $\gamma(\bar{k}+1)$ by $(c(\bar{k}), (\gamma(\bar{k}+2))_{-i})$. The new loop of profiles is connected to γ. Next, in the new loop of profiles, replace the $(\bar{k}+2)$th profile by $(c(\bar{k}), (\gamma(\bar{k}+3))_{-i})$. Again, the resulting loop of profiles is connected to the previous one, and hence connected to γ. Repeat this procedure until the $(k'+1)$th profile is replaced by $(c(\bar{k}), (\gamma(k'+2))_{-i})$.

Finally, delete the $(k'+1)$th profile, and insert $(c(\bar{k}), (\gamma(\bar{k}+1))_{-i})$ between the $\bar{k}$th and the $(\bar{k}+1)$th profiles. Let γ' be the resulting loop of profiles, and c' be the corresponding loop of agent i's preferences. By the construction of γ', $\gamma' \simeq \gamma$. For each k ($\bar{k} \leq k \leq k'+2$), $c'(k) = c'(\bar{k})$, and the value of $\tilde{w}(c', \cdot)$ is constant over this range. Also, for each k ($\bar{k} \leq k \leq k'$), $\tilde{w}(c', k) = \tilde{w}(c, k) - 1$. Moreover, $w(c') = w(c) = 0$, and for each k ($1 \leq k \leq K$) and each $j \neq i$, $(\gamma'(k))_j = (\gamma(k))_j$. If $\overline{\mathcal{K}}(c') = \emptyset$, then γ' is the desired loop γ^*. If $\overline{\mathcal{K}}(c') \neq \emptyset$, then let $\bar{k}, k' \in \overline{\mathcal{K}}(c')$ be such that $\alpha(c'(\bar{k}), c'(\bar{k}+1)) = 1$, $\alpha(c'(k), c'(k+1)) = 0$ for each k ($\bar{k}+1 \leq k \leq k'$), and $\alpha(c'(k'+1), c'(k'+2)) = -1$, and repeat the same argument as above. This procedure eventually finds the desired γ^*.

Next, if $\overline{W}(c^*) \geq 1$, by the same arguments, we can find $\gamma^{**} \in \Gamma$ with the corresponding loop c^{**} of agent i's preferences such that $\gamma^{**} \simeq \gamma^*$, $w(c^{**}) = 0$ $\overline{W}(c^{**}) = \overline{W}(c^*) - 1$, and $\underline{W}(c^{**}) = \underline{W}(c^*)$.

By repeating this procedure, we can find $\tilde{\gamma} \in \Gamma$ with the corresponding loop $\tilde{c}$ of agent i's preferences such that $\tilde{\gamma} \simeq \gamma$, $\overline{W}(\tilde{c}) = 0$, $\underline{W}(\tilde{c}) = \underline{W}(c)$, and $w(\tilde{c}) = 0$.

If $\underline{W}(\tilde{c}) \leq -1$, by applying symmetric arguments to $\tilde{\gamma}$, we can find $\tilde{\tilde{\gamma}} \in \Gamma$ with the corresponding loop $\tilde{\tilde{c}}$ of agent i's preferences such that $\tilde{\tilde{\gamma}} \simeq \tilde{\gamma} \simeq \gamma$, $\overline{W}(\tilde{\tilde{c}}) = \overline{W}(\tilde{c}) = 0$, and $\underline{W}(\tilde{\tilde{c}}) = 0$.

By the construction of $\tilde{\tilde{\gamma}}$, it is clear that $\tilde{\tilde{\gamma}} = \hat{\gamma}$. □

We next show that for each loop of preference profiles $\gamma \in \Gamma$, $\gamma \cdot \gamma^{-1} \simeq \gamma^0$. Let $\gamma = (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^K)$. Then

$$\begin{aligned}
\gamma \cdot \gamma^{-1} &= (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{K-2}, \boldsymbol{R}^{K-1}, \boldsymbol{R}^K, \boldsymbol{R}^1, \boldsymbol{R}^K, \boldsymbol{R}^{K-1}, \ldots, \boldsymbol{R}^2) \\
&\simeq (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{K-2}, \boldsymbol{R}^{K-1}, \boldsymbol{R}^K, \boldsymbol{R}^{K-1}, \ldots, \boldsymbol{R}^2) && \text{(by Lemma 3)} \\
&\simeq (\boldsymbol{R}^1, \ldots, \boldsymbol{R}^{K-2}, \boldsymbol{R}^{K-1}, \ldots, \boldsymbol{R}^2) && \text{(by Lemma 3)} \\
&\simeq \ldots \\
&\simeq (\boldsymbol{R}^1, \boldsymbol{R}^2) \\
&\simeq (\boldsymbol{R}^1, \boldsymbol{R}^1) && \text{(by (3) in Definition 2)} \\
&\simeq (\boldsymbol{R}^1) = \gamma^0.
\end{aligned}$$

Proof of Lemma 8. By Lemma 7, it suffices to establish $\bar{\gamma} \simeq \gamma_1^{\boldsymbol{R}} \cdot \gamma_2^{\boldsymbol{R}} \cdot \cdots \cdot \gamma_n^{\boldsymbol{R}}$ for some specific $\boldsymbol{R} \in \mathcal{L}^n$. In the following, let $\boldsymbol{R} \in \mathcal{L}^n$ be such that $R_i = \bar{R}^1$ for each $i \in N$.

Suppose that $\bar{\gamma}_{-i} \in \Gamma$ is a loop of preference profiles of length $6(n-1)$ such that agent i's preference is always $\bar{R}^1$ and preferences of each agent j ($\neq i$) changes according to $\bar{\gamma}$. By induction, it suffices to show that $\bar{\gamma} \simeq \gamma_1^{\boldsymbol{R}} \cdot \bar{\gamma}_{-1}$. Since $\bar{\gamma} \simeq \gamma_1^{\boldsymbol{R}} \cdot (\gamma_1^{\boldsymbol{R}})^{-1} \cdot \bar{\gamma}$, we want to show that $(\gamma_1^{\boldsymbol{R}})^{-1} \cdot \bar{\gamma} \simeq \bar{\gamma}_{-1}$, and moreover, in the process of deforming $(\gamma_1^{\boldsymbol{R}})^{-1} \cdot \bar{\gamma}$ to $\bar{\gamma}_{-1}$, the first profile remains the same.

Let c be the loop of agent i's preferences from $(\gamma_1^{\boldsymbol{R}})^{-1} \cdot \bar{\gamma}$. Since the loop of agent i's preferences from $(\gamma_1^{\boldsymbol{R}})^{-1}$ is $\bar{c}_{\bar{R}^1}^{-1}$, and the loop of agent i from $\bar{\gamma}$ is connected to $\bar{c}_{\bar{R}^1}$ by (2) in Definition 1, we have $w(c) = 0$. By Lemma 17, $(\gamma_1^{\boldsymbol{R}})^{-1} \cdot \bar{\gamma}$ is connected to the loop of preference profiles where every preference of agent 1 is replaced by $\bar{R}^1$. Then by (2) in Definition 2, we show that $(\gamma_1^{\boldsymbol{R}})^{-1} \cdot \bar{\gamma} \simeq \bar{\gamma}_{-1}$. This concludes the proof. □

A.3 Proof of Lemma 12

We first show preliminary lemmas. For each $R, R' \in \mathcal{L}$, let $s(R, R') = \min\{k \,|\, 1 \leq k \leq m, r^k(R) \neq r^k(R')\}$ if $R \neq R'$, and $s(R, R') = 0$ if $R = R'$. The following lemma provides a classification of adjacency relations when there are three preferences $R, R', R'' \in \mathcal{L}$ such that R' is adjacent or equal both to R and to R''.

Lemma 18. *For each pair of preferences $R, R'' \in \mathcal{L}$ ($R \neq R''$), let $\mathcal{A} \subseteq \mathcal{L}$ be the set of preferences that are adjacent or equal to R and also to R''. If $\mathcal{A} \neq \emptyset$, then one of the following three is true:*

(a) R and R'' are adjacent, and $\mathcal{A} = \{R, R''\}$,

(b) R and R'' are not adjacent, $\mathcal{A} \cap \{R, R''\} = \emptyset$, and $\mathcal{A} = \{R'\}$ where $R' \in \mathcal{L}$ satisfies $|s(R, R') - s(R', R'')| = 1$, or

(c) R and R'' are not adjacent, $\mathcal{A} \cap \{R, R''\} = \emptyset$, $|\mathcal{A}| = 2$, and each $R' \in \mathcal{A}$ satisfies $|s(R, R') - s(R', R'')| \geq 2$.

Proof of Lemma 18. If R and R'' are adjacent, then $\mathcal{A} = \{R, R''\}$ is immediate. This is case (a). If R and R'' are not adjacent, then $\mathcal{A} \cap \{R, R''\} = \emptyset$ is obvious.

Suppose that there exists $R' \in \mathcal{L}$ such that R and R' are adjacent, and R' and R'' are adjacent. Since $R \neq R''$, $s(R, R') \neq s(R', R'')$. First, suppose that $|s(R, R') - s(R', R'')| = 1$. By symmetry of the adjacency relation, we can assume without loss of generality that $s(R, R') = s(R', R'') - 1$. Let $\bar{k} = s(R, R')$. Then $r^{\bar{k}}(R) = r^{\bar{k}+1}(R') = r^{\bar{k}+2}(R'')$, $r^{\bar{k}+1}(R) = r^{\bar{k}}(R') = r^{\bar{k}}(R'')$, $r^{\bar{k}+2}(R) = r^{\bar{k}+2}(R') = r^{\bar{k}+1}(R'')$, and for each $k \neq \bar{k}, \bar{k}+1, \bar{k}+2$, $r^k(R) = r^k(R') = r^k(R'')$. It is easy to see that no other preference is adjacent to both R and to R''. This is case (b).

Second, suppose that $|s(R, R') - s(R', R'')| \geq 2$. Let $\bar{k} = s(R, R')$ and $\bar{k}' = s(R', R'')$. Then $r^{\bar{k}}(R) = r^{\bar{k}+1}(R') = r^{\bar{k}+1}(R'')$, $r^{\bar{k}+1}(R) = r^{\bar{k}}(R') = r^{\bar{k}}(R'')$, $r^{\bar{k}'}(R) = r^{\bar{k}'}(R') = r^{\bar{k}'+1}(R'')$, $r^{\bar{k}'+1}(R) = r^{\bar{k}'+1}(R') = r^{\bar{k}'}(R'')$, and for each $k \neq \bar{k}, \bar{k}+1, \bar{k}', \bar{k}'+1$, $r^k(R) = r^k(R') = r^k(R'')$. Let use define a preference $\bar{R}' \in \mathcal{L}$ by $r^{\bar{k}}(\bar{R}') = r^{\bar{k}+1}(R')$, $r^{\bar{k}+1}(\bar{R}') = r^{\bar{k}}(R')$, $r^{\bar{k}'}(\bar{R}') = r^{\bar{k}'+1}(R')$, $r^{\bar{k}'+1}(\bar{R}') = r^{\bar{k}'}(R')$, and for each $k \neq \bar{k}, \bar{k}+1, \bar{k}', \bar{k}'+1$, $r^k(\bar{R}') = r^k(R')$. It is easy to see $\mathcal{A} = \{R', \bar{R}'\}$. This is case (c). □

The following lemma considers loops $c \in C$ with length 6, and claims that if c is not connected to a trivial loop with length 1, then the adjacency relations in c must be of the type (b) in Lemma 18, i.e., for each k ($1 \leq k \leq 6$), $c(k+1)$ is the unique preference that is adjacent or equal to both $c(k)$ and $c(k+2)$.

Lemma 19. *For each $c \in C$ such that* $\text{length}(c) = 6$ *and $c \not\simeq \big(c(1)\big)$, and for each k ($1 \leq k \leq 6$), $c(k) \neq c(k+2)$ and $c(k)$ and $c(k+2)$ satisfy condition (b) in Lemma 18.*

Proof of Lemma 19. We first show that if $\text{length}(c) \leq 5$, then $c \simeq \big(c(1)\big)$. For each $R, R' \in \mathcal{L}$ such that R and R' are adjacent, there exist y and z such that $y \; R \; z$ and $z \; R' \; y$. Therefore if $\text{length}(c) = 5$, there exists k such that $c(k) = c(k+1)$. We can assume that $\text{length}(c) \leq 4$ by condition (2) in Definition 1. Suppose that $\text{length}(c) = 4$. By condition (3) in Definition 1, $c \simeq \big(c(1), c(2), c(3), c(2)\big)$, which is connected to $\big(c(1)\big)$ by Lemma 3 and conditions (2) and (3) in Definition 1. By similar arguments, we can show that $c \simeq \big(c(1)\big)$ whenever $\text{length}(c) \leq 3$. Hence, if $\text{length}(c) \leq 5$, then $c \simeq \big(c(1)\big)$.

Let $c \in C$ be such that $\text{length}(c) = 6$ and $c \not\simeq \big(c(1)\big)$. First, we claim $c(k) \neq c(k+2)$ for each k ($1 \leq k \leq 6$). Suppose that there exists k such that $c(k) = c(k+2)$. By conditions (2) and (3) in Definition 1 and Lemma 2, $c \simeq \big(c(k), c(k+2), c(k+3), c(k+4), c(k+5)\big)$ which is a loop of length 5. Therefore, $c \simeq \big(c(1)\big)$, which is a contradiction. Thus, $c(k) \neq c(k+2)$ for each k ($1 \leq k \leq 6$).

Next, we show that $c(k)$ and $c(k+2)$ satisfy condition (b) in Lemma 18. Suppose that there exists k such that $c(k)$ and $c(k+2)$ do not satisfy condition (b) in Lemma 18. By Lemma 18, $c(k)$ and $c(k+2)$ satisfy either condition (a) or (c) in Lemma 18.

Case 1: Suppose that $c(k)$ and $c(k+2)$ satisfy condition (a) in Lemma 18. Then either $c(k) = c(k+1)$ or $c(k+1) = c(k+2)$. By condition (2) in Definition 1 and Lemma 2, $c \simeq \big(c(k), c(k+2), c(k+3), c(k+4), c(k+5)\big)$ which is a loop of length 5. Therefore, $c \simeq \big(c(1)\big)$.

Case 2: Suppose that $c(k)$ and $c(k+2)$ satisfy condition (c) in Lemma 18. By Lemma 2, we assume without loss of generality that $c(1)$ and $c(3)$ satisfy condition (c) in Lemma 18. Let $\bar{k} = s\big(c(1), c(2)\big)$, $\bar{k}' = s\big(c(2), c(3)\big)$, and $\bar{k}'' = s\big(c(3), c(4)\big)$. Since $c(1)$ and $c(3)$ satisfy condition (c), $|\bar{k} - \bar{k}'| \geq 2$. Let $x = r^{\bar{k}}(c(1))$, $y = r^{\bar{k}+1}(c(1))$, $x' = r^{\bar{k}'}(c(1))$, and $y' = r^{\bar{k}'+1}(c(1))$. Since $|\bar{k} - \bar{k}'| \geq 2$, these four alternatives are all distinct from each other.

Since $c(2) \neq c(4)$, by Lemma 18, $c(2)$ and $c(4)$ satisfy one of conditions (a), (b), and (c) in Lemma 18. If $c(2)$ and $c(4)$ satisfy condition (a) in Lemma 18, then Case 1 applies and we have $c \simeq \big(c(1)\big)$, which is a contradiction. Let us consider the remaining two subcases.

Case 2.1: Suppose that $c(2)$ and $c(4)$ satisfy condition (b) in Lemma 18. Then $|\bar{k}' - \bar{k}''| = 1$. First, assume that $|\bar{k} - \bar{k}''| = 1$. We claim $|\bar{k} - \bar{k}'| = 2$. By assumption, $|\bar{k} - \bar{k}'| \geq 2$.

rank	$c(1)$	$c(2)$	$c(3)$	$c(4)$	$c(1)$	$c(2)$	$c(3)$	$c(4)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}-2$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	x'	x'	y'	y'
$\bar{k}-1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	y'	y'	x'	y
$\bar{k}$	x	y	y	y	x	y	y	x'
$\bar{k}+1$	y	x	x	y'	y	x	x	x
$\bar{k}+2$	x'	x'	y'	x	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}+3$	y'	y'	x'	x'	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3: Case 2.1 when $|\bar{k}-\bar{k}''|=1$

When $\bar{k}>\bar{k}'$, $\bar{k}-\bar{k}'\geq 2$. Then, for $\bar{k}''$ to be $|\bar{k}-\bar{k}''|=1$ and $|\bar{k}'-\bar{k}''|=1$, $\bar{k}''$ should be between $\bar{k}$ and $\bar{k}'$ (i.e. $\bar{k}>\bar{k}''>\bar{k}'$) and $|\bar{k}-\bar{k}'|=2$. This case is described by the left half of Table 3. Similar arguments apply when $\bar{k}<\bar{k}'$, and the situation is described by the right half of Table 3. In the former case, one of the pairs (x,y), (x',y'), and (x,y') must be exchanged between $c(4)$ and $c(5)$, or c cannot be a loop. Since the only consecutively ranked pair in $c(4)$ is (x,y'), we have $c(5)=c(3)$, which is a contradiction. In the latter case, one of the pairs (x,y), (x',y'), and (x',y) must be exchanged between $c(4)$ and $c(5)$. Since the only consecutively ranked pair in $c(4)$ is (x',y), we have $c(5)=c(3)$, which is a contradiction.

Second, assume that $|\bar{k}-\bar{k}''|\geq 2$. Since $|\bar{k}-\bar{k}'|\geq 2$, $|\bar{k}-\bar{k}''|\geq 2$, and $|\bar{k}'-\bar{k}''|=1$, $\bar{k}$ cannot be between $\bar{k}'$ and $\bar{k}''$. Thus, we have one of the following: $[\bar{k}<\bar{k}'<\bar{k}'']$, $[\bar{k}<\bar{k}''<\bar{k}']$, $[\bar{k}'<\bar{k}''<\bar{k}]$, or $[\bar{k}''<\bar{k}'<\bar{k}]$. (Brackets are just for readability.) We consider only the first case which is shown in Table 4. (Similar arguments apply to the remaining cases.) Since c is a loop, there exists unique k ($4\leq k\leq 6$) such that $s\big(c(k),c(k+1)\big)=\bar{k}$. If $k=4$ or $k=5$, then x and y are not exchanged between $c(6)$ and $c(1)$. This implies that x' and y' are exchanged between $c(6)$ and $c(1)$. Therefore, the only difference between $c(6)$ and $c(1)$ is the positions of x' and y'. Then, $c(3)$ and $c(6)$ are adjacent, and $c\simeq\big(c(1),c(2),c(3),c(6)\big)$ by Lemma 3. Since $\text{length}\big(c(1),c(2),c(3),c(6)\big)\leq 5$, $c\simeq\big(c(1)\big)$, which is a contradiction. If $k=6$, then x and y are not exchanged between $c(4)$ and $c(5)$. This implies $c(5)=c(3)$, which is also a contradiction.

Case 2.2: Suppose that $c(2)$ and $c(4)$ satisfy condition (c) in Lemma 18. Then $|\bar{k}'-\bar{k}''|\geq 2$. First, suppose that $|\bar{k}-\bar{k}''|\leq 1$. Let $R'\neq c(2)$ be a preference which is adjacent to both $c(1)$ and $c(3)$. Such R' exists uniquely because $c(1)$ and $c(3)$ satisfy condition (c)

rank	$c(1)$	$c(2)$	$c(3)$	$c(4)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}$	x	y	y	y
$\bar{k}+1$	y	x	x	x
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}'$	x'	x'	y'	y'
$\bar{k}'+1$	y'	y'	x'	y''
$\bar{k}'+2$	y''	y''	y''	x'
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4: An example of Case 2.1 when $|\bar{k}-\bar{k}''| \geq 2$

rank	$c(1)$	$c(2)$	$c(3)$	$c(4)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}$	x	y	y	y
$\bar{k}+1$	y	x	x	x
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}'$	x'	x'	y'	y'
$\bar{k}'+1$	y'	y'	x'	x'
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}''$	x''	x''	x''	y''
$\bar{k}'+1$	y''	y''	y''	x''
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 5: An example of Case 2.2 when $|\bar{k}-\bar{k}''| \geq 2$

in Lemma 18. By condition (3) in Definition 1, $\big(c(1), R', c(3), c(4), c(5), c(6)\big)$ is a loop connected to c. If $\bar{k} = \bar{k''}$, then $R' = c(4)$, which is a contradiction. If $|\bar{k} - \bar{k''}| = 1$, then R' and $c(4)$ satisfy condition (b) in Lemma 18. We have a contradiction by applying arguments of the case $|\bar{k} - \bar{k''}| \geq 2$ in Case 2.1 to the loop $\big(c(1), R', c(3), c(4), c(5), c(6)\big)$.

Second, suppose that $|\bar{k} - \bar{k}''| \geq 2$. Let $x'' = r^{\bar{k}''}(c(1))$, and $y'' = r^{\bar{k}''+1}(c(1))$. There are six possible ways of ordering $\bar{k}$, $\bar{k}'$, and $\bar{k''}$. We consider only the case $\bar{k} < \bar{k}' < \bar{k''}$ as presented in Table 5. (Similar arguments apply to the remaining cases.) Since c is a loop, one of the pairs (x, y), (x', y'), and (x'', y'') must be exchanged between $c(4)$ and $c(5)$. If x'' and y'' are exchanged between $c(4)$ and $c(5)$, then $c(3) = c(5)$, which is a contradiction. If x' and y' are exchanged between $c(4)$ and $c(5)$, then $c(2)$ is adjacent to both $c(3)$ and $c(5)$, and thus $c \simeq (c(1), c(2), c(3), c(2), c(5), c(6)) := c'$. Since $c'(2) = c'(4)$,

rank	R	R'	$\bar{R}^{l,t}$	$\bar{R}^{l+1,t}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$s(R,R')$	x	y	x	y
$s(R,R')+1$	y	x	y	x
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 6: The case of $\alpha_t(R,R')=1$.

by similar arguments before Case 1, $c' \simeq (c'(1)) = (c(1))$. Therefore, $c \simeq (c(1))$, which is a contradiction. If x and y are exchanged between $c(4)$ and $c(5)$, then there exists a unique preference $R' \neq c(4)$ such that R' is adjacent to both $c(3)$ and $c(5)$, and $s(c(3),R') = \bar{k}$. By condition (3) in Definition 1, $(c(1),c(2),c(3),R',c(5),c(6))$ is a loop connected to c, and the first case in Case 2.2 shows this case.

Hence, we obtain the lemma. □

We now introduce $w_t(c)$ for each $c \in C$ and each t ($1 \leq t \leq m-2$), which is a generalization of the function w defined in Section 3. For each $R, R' \in \mathcal{L}$ and each t ($1 \leq t \leq m-2$), let $\alpha_t(R,R') \in \{-1,0,1\}$ be defined as follows:

(1) $\alpha_t(R,R') = 1$ if R and R' are adjacent, there exists $(l,l') \in \{(1,2),(2,3),(3,4),(4,5),(5,6),(6,1)\}$ such that $s(R,R') = s(\bar{R}^{l,t},\bar{R}^{l',t})$, and $r^k(R) = r^k(\bar{R}^{l,t})$ for each $k = s(R,R')$ and $s(R,R')+1$.

(2) $\alpha_t(R,R') = -1$ if R and R' are adjacent, there exists $(l,l') \in \{(2,1),(3,2),(4,3),(5,4),(6,5),(1,6)\}$ such that $s(R,R') = s(\bar{R}^{l,t},\bar{R}^{l',t})$, and $r^k(R) = r^k(\bar{R}^{l,t})$ for each $k = s(R,R')$ and $s(R,R')+1$.

(3) $\alpha_t(R,R') = 0$ otherwise.

The first case is summarized in Table 6. Let x and y be the consecutively ranked alternatives whose ranks are exchanged in the passage from R to R'. Then, the rank of x in R is $s(R,R')$. Since the $s(R,R')$th ranked alternatives in R and $\bar{R}^{l,t}$ should be the same, x is also $s(R,R')$th ranked in $\bar{R}^{l,t}$. Also, the $(s(R,R')+1)$th ranked alternatives in R and $\bar{R}^{l,t}$ should be the same, and y is $(s(R,R')+1)$th ranked in $\bar{R}^{l,t}$. Since $s(R,R') = s(\bar{R}^{l,t},\bar{R}^{l+1,t})$, the ranks of x and y should be exchanged in the passage from $\bar{R}^{l,t}$ to $\bar{R}^{l+1,t}$. Since $\bar{R}^{l,t}$ and $\bar{R}^{l+1,t}$ are adjacent, there are no other differences between $\bar{R}^{l,t}$ and $\bar{R}^{l+1,t}$.

For each $c \in C$ and each t ($1 \leq t \leq m-2$), we define $w_t(c) \in \mathbb{R}$ by

$$w_t(c) = \frac{1}{6} \sum_{k=1}^{\text{length}(c)} \alpha_t\big(c(k), c(k+1)\big). \tag{4}$$

We note that if $m = 3$, this definition coincides with that in Section 3. We have the following lemma which is a generalization of a part of Lemma 5.

Lemma 20. *For each $c, c' \in C$ and each t ($1 \leq t \leq m-2$), if $c \simeq c'$, then $w_t(c) = w_t(c')$.*

Proof of Lemma 20. Fix t ($1 \leq t \leq m-2$) arbitrarily. It suffices to prove $w_t(c) = w_t(c')$ whenever c and c' satisfy either of conditions (1), (2), or (3) in Definition 1. If c and c' satisfy condition (1) or (2), then the same argument as in the proof of Lemma 16 shows $w_t(c) = w_t(c')$.

Suppose that c and c' satisfy (3) in Definition 1. Then $\text{length}(c') = \text{length}(c)$, and there exists $\bar{k}$ ($1 \leq \bar{k} \leq \text{length}(c)$) such that $c'(k) = c(k)$ whenever $k \neq \bar{k}$. Since $\alpha_t\big(c'(k), c'(k+1)\big) = \alpha_t\big(c(k), c(k+1)\big)$ whenever $k \neq \bar{k}-1, \bar{k}$, we have

$$\begin{aligned} &w_t(c) - w_t(c') \\ &\quad = \alpha_t\big(c(\bar{k}-1), c(\bar{k})\big) + \alpha_t\big(c(\bar{k}), c(\bar{k}+1)\big) - \alpha_t\big(c'(\bar{k}-1), c'(\bar{k})\big) - \alpha_t\big(c'(\bar{k}), c'(\bar{k}+1)\big). \end{aligned}$$

Let $R = c(\bar{k}-1) = c'(\bar{k}-1)$ and $R'' = c(\bar{k}+1) = c'(\bar{k}+1)$. If $R = R''$, it is clear that $w_t(c) = w_t(c')$. We consider three cases in Lemma 18 in order.

First, suppose that case (a) in Lemma 18 holds. Let $\tilde{c} = \big(c(1), \ldots, c(\bar{k}-1), c(\bar{k}+1), \ldots, c(\text{length}(c))\big)$. Since $w_t(c) = w_t(\tilde{c})$ and $w_t(c') = w_t(\tilde{c})$, we have $w_t(c) = w_t(c')$.

Second, suppose that case (b) in Lemma 18 holds. Since there exists a unique preference $R' \in \mathcal{L}$ such that R' is adjacent or equal to both R and R'', we have $c(\bar{k}) = c'(\bar{k}) = R'$. Therefore $c = c'$, and clearly $w_t(c) = w_t(c')$.

Third, suppose that case (c) in Lemma 18 holds. By the proof of Lemma 18, if $c(\bar{k}) \neq c'(\bar{k})$, then $s\big(c(\bar{k}-1), c(\bar{k})\big) = s\big(c'(\bar{k}), c'(\bar{k}+1)\big)$ and $s\big(c(\bar{k}), c(\bar{k}+1)\big) = s\big(c'(\bar{k}-1), c'(\bar{k})\big)$. By the definition of α_t, this implies that $\alpha_t\big(c(\bar{k}-1), c(\bar{k})\big) = \alpha_t\big(c'(\bar{k}), c'(\bar{k}+1)\big)$ and $\alpha_t\big(c(\bar{k}), c(\bar{k}+1)\big) = \alpha_t\big(c'(\bar{k}-1), c'(\bar{k})\big)$. Hence we have $w_t(c) = w_t(c')$. □

Proof of Lemma 12. We first prove (a). We assume that $\text{length}(c) = 6$, $c(1) = \bar{R}$, and $c \not\simeq (\bar{R})$, and show that there exists t ($1 \leq t \leq m-2$) such that $c = \bar{c}_{\bar{R},t}$ or $(\bar{c}_{\bar{R},t})^{-1}$. Fix k^* ($1 \leq k^* \leq 6$), and let $\bar{k} = s(c(k^*), c(k^*+1))$, $\bar{k}' = s(c(k^*+1), c(k^*+2))$, and $\bar{k}'' = s(c(k^*+2), c(k^*+3))$. By Lemma 19, $c(k) \neq c(k+2)$, and $c(k)$ and $c(k+2)$ satisfy condition (b) in Lemma 18 for each k ($1 \leq k \leq 6$), and thus $|\bar{k} - \bar{k}'| = |\bar{k}' - \bar{k}''| = 1$. This implies that $\bar{k} = \bar{k}''$ or $|\bar{k} - \bar{k}''| = 2$. We can show $\bar{k} = \bar{k}''$.

rank	$c(k^*)$	$c(k^*+1)$	$c(k^*+2)$	$c(k^*+3)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{k}$	x	y	y	y
$\bar{k}+1$	y	x	y'	y'
$\bar{k}+2$	y'	y'	x	y''
$\bar{k}+3$	y''	y''	y''	x
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 7: An example when $|\bar{k} - \bar{k}''| = 2$

Assume that $|\bar{k} - \bar{k}''| = 2$. We have either $\bar{k} < \bar{k}''$ or $\bar{k} > \bar{k}''$. We consider only the former case as described in Table 7. (Similar arguments apply to the latter case.) Since c is a loop, x and y'' must be exchanged between $c(k^*+3)$ and $c(k^*+4)$ and thus $c(k^*+2) = c(k^*+4)$, which is a contradiction. Hence, we must have $\bar{k} = \bar{k}''$.

Let $\bar{k}^1 = s(c(1), c(2))$, $\bar{k}^2 = s(c(2), c(3))$, $\ldots$, $\bar{k}^6 = s(c(6), c(1))$. Since k^* was arbitrary, $\bar{k}^1 = \bar{k}^3 = \bar{k}^5$, $\bar{k}^2 = \bar{k}^4 = \bar{k}^6$, and $|\bar{k}^1 - \bar{k}^2| = 1$. When $\bar{k}^1 < \bar{k}^2$, $c(k) \in \mathcal{D}_{\bar{k}^1}$ for each k ($1 \le k \le 6$). When $\bar{k}^1 > \bar{k}^2$, $c(k) \in \mathcal{D}_{\bar{k}^2}$ for each k ($1 \le k \le 6$). Therefore, there exists t ($1 \le t \le m-2$) such that $c(k) \in \mathcal{D}_t$ for each k ($1 \le k \le 6$). Then the same argument as in Lemmas 5 and 6 show that $w_t(c)$ is an integer, and (i) if $w_t(c) = 0$ then $c \simeq (\bar{R})$, (ii) if $w_t(c) = 1$ then $c = \bar{c}_{\bar{R},t}$, and (iii) if $w_t(c) = -1$ then $c = (\bar{c}_{\bar{R},t})^{-1}$. This proves (a) in the lemma.

We next prove (b). Let $c \in C$ be a loop such that $c(1) \in \mathcal{D}_t$ and $c \simeq \bar{c}_{\bar{R},t}$. Let $c(1) = \bar{R}^{l,t}$. By Lemma 20, $w_t(c) = 1$. Thus, for each k ($1 \le k \le 6$), $\alpha_t\big(c(k), c(k+1)\big) = 1$. For $\alpha_t\big(c(1), c(2)\big) = 1$, $s\big(c(1), c(2)\big)$ is either t or $t+1$. In either case, $c(2) \in \mathcal{D}_t$. (See Table 2.) Thus, $c(2)$ is either $\bar{R}^{l-1,t}$ or $\bar{R}^{l+1,t}$. Since $\alpha_t\big(c(1), c(2)\big) = 1$, we have $c(2) = \bar{R}^{l+1,t}$. By similar arguments, we have $c(3) = \bar{R}^{l+2,t}$, $c(4) = \bar{R}^{l+3,t}$, $c(5) = \bar{R}^{l+4,t}$, and $c(6) = \bar{R}^{l+5,t}$. Therefore, $c = \bar{c}_{c(1),t}$. $\square$

A.4 Proof of Lemma 13

Fix t ($1 \le t \le m-2$) arbitrary. By *adjacency-extended unanimity*, for each $i \in N$, $f(\gamma_i^{\bar{\mathbf{R}},t})(1) = \bar{R} \in \mathcal{D}_t$, $f(\gamma_i^{\bar{\mathbf{R}},t})(2) \in \{\bar{R}, \bar{R}^{2,t}\}$, and $f(\gamma_i^{\bar{\mathbf{R}},t})(6) \in \{\bar{R}, \bar{R}^{6,t}\}$. By Lemma 12 (a), either $f(\gamma_i^{\bar{\mathbf{R}},t}) \simeq (\bar{R})$, or there exists t^* ($1 \le t^* \le m-2$) such that $f(\gamma_i^{\bar{\mathbf{R}},t}) = \bar{c}_{\bar{R},t^*}$ or $\big(\bar{c}_{\bar{R},t^*}\big)^{-1}$. If $f(\gamma_i^{\bar{\mathbf{R}},t}) = \bar{c}_{\bar{R},t^*}$, the loop $\bar{c}_{\bar{R},t^*}$ should contain $\bar{R}$, $\bar{R}^{2,t}$, and $\bar{R}^{6,t}$. This implies $t^* = t$. The same argument shows $t^* = t$ when $f(\gamma_i^{\bar{\mathbf{R}},t}) = \big(\bar{c}_{\bar{R},t^*}\big)^{-1}$. Moreover, since $f(\gamma_i^{\bar{\mathbf{R}},t})(2) \in \{\bar{R}, \bar{R}^{2,t}\}$, either $f(\gamma_i^{\bar{\mathbf{R}},t}) \simeq (\bar{R})$ or $f(\gamma_i^{\bar{\mathbf{R}},t}) = \bar{c}_{\bar{R},t}$.

By Lemma 8, $\gamma_1^{\bar{\boldsymbol{R}},t} \cdot \cdots \cdot \gamma_n^{\bar{\boldsymbol{R}},t} \simeq \bar{\gamma}$, where $\bar{\gamma} \in \Gamma$ is defined similarly to that before Lemma 7. By Lemma 4, $f(\gamma_1^{\bar{\boldsymbol{R}},t}) \cdot \cdots \cdot f(\gamma_n^{\bar{\boldsymbol{R}},t}) \simeq f(\bar{\gamma})$. By *adjacency-extended unanimity*, $f(\bar{\gamma}) \simeq \bar{c}_{\bar{R},t}$. Thus

$$f(\gamma_1^{\bar{\boldsymbol{R}},t}) \cdot \cdots \cdot f(\gamma_n^{\bar{\boldsymbol{R}},t}) \simeq \bar{c}_{\bar{R},t}.$$

By Lemma 20,

$$w_t\big(f(\gamma_1^{\bar{\boldsymbol{R}},t})\big) + \cdots + w_t\big(f(\gamma_n^{\bar{\boldsymbol{R}},t})\big) = 1.$$

Since either $f(\gamma_i^{\bar{\boldsymbol{R}},t}) \simeq (\bar{R})$ or $f(\gamma_i^{\bar{\boldsymbol{R}},t}) = \bar{c}_{\bar{R},t}$ for each $i \in N$, there exists a unique agent $i_t \in N$ such that $f(\gamma_{i_t}^{\bar{\boldsymbol{R}},t}) = \bar{c}_{\bar{R},t}$ and $f(\gamma_j^{\bar{\boldsymbol{R}},t}) \simeq (\bar{R})$ for each $j \neq i_t$. By Lemmas 4 and 7, $f(\gamma_{i_t}^{\boldsymbol{R},t}) \simeq \bar{c}_{\bar{R},t}$ for and each $\boldsymbol{R} \in \mathcal{L}^n$ such that $R_i \in \mathcal{D}_t$, and by Lemma 12 (b), we have $f(\gamma_{i_t}^{\boldsymbol{R},t}) = \bar{c}_{f(\boldsymbol{R}),t}$ if $f(\boldsymbol{R}) \in \mathcal{D}_t$.

Since t was arbitrary, there is i_t for each $t (1 \leq t \leq m-2)$. In the following, we show that they are the identical agent. For a fixed t' $(1 \leq t' \leq m-3)$, assume that $i_{t'} \neq i_{t'+1}$. Since $f(\gamma_{i_{t'+1}}^{\bar{\boldsymbol{R}},t'+1}) = \bar{c}_{\bar{R},t'+1}$, $f(\bar{R}^{6,t'+1}, \bar{\boldsymbol{R}}_{-i_{t'+1}}) = \bar{c}_{\bar{R},t'+1}(6) = \bar{R}^{6,t'+1} = \bar{R}^{2,t'} \in \mathcal{D}_{t'}$. Then, $f\big(\bar{R}^{2,t'}, \bar{R}^{6,t'+1}, \bar{\boldsymbol{R}}_{-(i_{t'},i_{t'+1})}\big) = f\big(\gamma_{i_{t'}}^{(\bar{R}^{6,t'+1},\bar{\boldsymbol{R}}_{-i_{t'+1}})}\big)(2) = \bar{c}_{\bar{R}^{2,t'},t'}(2) = \bar{R}^{3,t'} \notin \{\bar{R}, \bar{R}^{2,t'}\}$. This contradicts *adjacency-extended unanimity* because $\bar{R}$ and $\bar{R}^{2,t'}$ $(= \bar{R}^{6,t'+1})$ are adjacent. Therefore we have $i_{t'} = i_{t'+1}$ for each t' $(1 \leq t' \leq m-3)$. This completes the proof.

A.5 Proof of Proposition 15

The definitions of loops and the relation $\simeq$ between two loops in $\mathcal{R}$ in Section 2 are generalized to $\mathcal{R}$ in an obvious manner with respect to the adjacency relation introduced in Section 5.

For each $R \in \mathcal{R}$ and each k $(1 \leq k \leq m)$, let $\hat{r}^k(R) = \big\{x \in X \,|\, \#\{y \in X \,|\, x \ R \ y\} = m-k+1\big\}$ be the set of kth-ranked alternatives at R. For each $R, R' \in \mathcal{R}$, let $\hat{s}(R, R') = \min\{k \,|\, 1 \leq k \leq m, \hat{r}^k(R) \neq \hat{r}^k(R')\}$ if $R \neq R'$, and $\hat{s}(R, R') = 0$ if $R = R'$.

Let us fix a preference $\bar{R} \in \mathcal{L}$ arbitrarily. For each t $(1 \leq t \leq m-2)$, we introduce a subset $\hat{\mathcal{D}}_t$ of $\mathcal{R}$. For each t $(1 \leq t \leq m-2)$, the domain $\hat{\mathcal{D}}_t$ comprises twelve preferences $\hat{R}^{1,t}, \ldots, \hat{R}^{12,t}$ presented in Table 8, in which $\hat{R}^{1,t} = \bar{R}$ for each t.[6] By rule, for each integer l', let $\hat{R}^{l',t} = \hat{R}^{l,t}$ if $l' = 12p + l$ for some integer p. In this notation, $\hat{R}^{l,t}$ and $\hat{R}^{l',t}$ are adjacent if and only if there exists an integer p such that (a) $l - l' = 12p + 1$ or $12p - 1$, or (b) $l - l' = 12p + 2$ or $12p - 2$, and l is odd. We note that for each t $(1 \leq t \leq m-3)$, $\hat{R}^{2,t} = \hat{R}^{12,t+1}$, $\hat{R}^{3,t} = \hat{R}^{11,t+1}$, and $\hat{\mathcal{D}}_t \cap \hat{\mathcal{D}}_{t+1} = \{\bar{R}, \hat{R}^{2,t}, \hat{R}^{3,t}\}$. Thus for each t $(1 \leq t \leq m-3)$, $\hat{\mathcal{D}}_t \cup \hat{\mathcal{D}}_{t+1}$ comprises 21 preferences presented in the graph in Figure 5, in which

[6]If two alternative are put in the same cell, these are indifferent under the preference defined by the column.

$\hat{R}^{1,t}$	$\hat{R}^{2,t}$	$\hat{R}^{3,t}$	$\hat{R}^{4,t}$	$\hat{R}^{5,t}$	$\hat{R}^{6,t}$
x^1	x^1	x^1	x^1	x^1	x^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}
x^t	x^t	x^t	x^t, x^{t+2}	x^{t+2}	x^{t+2}
x^{t+1}	x^{t+1}, x^{t+2}	x^{t+2}		x^t	x^t, x^{t+1}
x^{t+2}		x^{t+1}	x^{t+1}	x^{t+1}	
x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^m	x^m	x^m	x^m	x^m	x^m

$\hat{R}^{7,t}$	$\hat{R}^{8,t}$	$\hat{R}^{9,t}$	$\hat{R}^{10,t}$	$\hat{R}^{11,t}$	$\hat{R}^{12,t}$
x^1	x^1	x^1	x^1	x^1	x^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}	x^{t-1}
x^{t+2}	x^{t+1}, x^{t+2}	x^{t+1}	x^{t+1}	x^{t+1}	x^t, x^{t+1}
x^{t+1}		x^{t+2}	x^t, x^{t+2}	x^t	
x^t	x^t	x^t		x^{t+2}	x^{t+2}
x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}	x^{t+3}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^m	x^m	x^m	x^m	x^m	x^m

Table 8: Preferences in $\hat{\mathcal{D}}_t$. We note that $\hat{R}^{1,t} = \bar{R}$ for each t ($1 \leq t \leq m-2$).

each pair of adjacent preferences is connected with an edge. We note that for each t ($1 \leq t \leq m-2$), $\hat{R}^{l,t} \in \mathcal{L}$ if and only if l is odd, and if l is odd, $\hat{R}^{l,t} = \bar{R}^{\frac{l+1}{2},t}$ where $\bar{R}^{l',t}$ is the notation introduced in Section 4.

We introduce $w_t(c)$ for each loop c in $\mathcal{R}$ and each t ($1 \leq t \leq m-2$), which is an extension of the function w_t defined in Appendix A.3. For each $R, R' \in \mathcal{R}$ and each t ($1 \leq t \leq m-2$), let $\alpha_t(R, R') \in \{-1, -1/2, 0, 1/2, 1\}$ be defined as follows:

(1) $\alpha_t(R, R') = 1$ if preferences R and R' are adjacent, there exists a pair of integers $(l, l') \in \{(1,3), (3,5), (5,7), (7,9), (9,11), (11,1)\}$ such that $\hat{s}(R, R') = \hat{s}(\hat{R}^{l,t}, \hat{R}^{l',t})$, and $\hat{r}^k(R) = \hat{r}^k(\hat{R}^{l,t})$ and $\hat{r}^k(R') = \hat{r}^k(\hat{R}^{l',t})$ for each $k = \hat{s}(R, R')$ and $\hat{s}(R, R') + 1$.

(2) $\alpha_t(R, R') = \frac{1}{2}$ if preferences R and R' are adjacent, there exists a pair of integers $(l, l') \in \{(1,2), (2,3), (3,4), \ldots, (11,12), (12,1)\}$ such that $\hat{s}(R, R') = \hat{s}(\hat{R}^{l,t}, \hat{R}^{l',t})$, and $\hat{r}^k(R) = \hat{r}^k(\hat{R}^{l,t})$ and $\hat{r}^k(R') = \hat{r}^k(\hat{R}^{l',t})$ for each $k = \hat{s}(R, R')$ and $\hat{s}(R, R') + 1$.

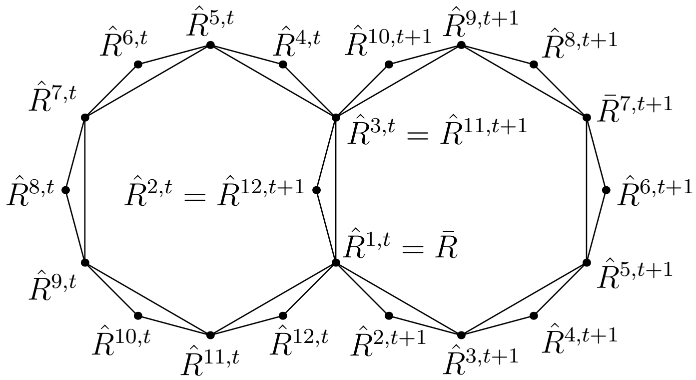

Figure 5: A graph representing the adjacent relations in the restricted domain $\hat{\mathcal{D}}_t \cup \hat{\mathcal{D}}_{t+1}$.

(3) $\alpha_t(R, R') = -1$ if preferences R and R' are adjacent, there exists a pair of integers $(l, l') \in \{(3,1), (5,3), (7,5), (9,7), (11,9), (1,11)\}$ such that $\hat{s}(R, R') = \hat{s}(\hat{R}^{l,t}, \hat{R}^{l',t})$, and $\hat{r}^k(R) = \hat{r}^k(\hat{R}^{l,t})$ and $\hat{r}^k(R') = \hat{r}^k(\hat{R}^{l',t})$ for each $k = \hat{s}(R, R')$ and $\hat{s}(R, R') + 1$.

(4) $\alpha_t(R, R') = -\frac{1}{2}$ if preferences R and R' are adjacent, there exists a pair of integers $(l, l') \in \{(2,1), (3,2), (4,3), \ldots, (12,11), (1,12)\}$ such that $\hat{s}(R, R') = \hat{s}(\hat{R}^{l,t}, \hat{R}^{l',t})$, and $\hat{r}^k(R) = \hat{r}^k(\hat{R}^{l,t})$ and $\hat{r}^k(R') = \hat{r}^k(\hat{R}^{l',t})$ for each $k = \hat{s}(R, R')$ and $\hat{s}(R, R') + 1$.

(5) $\alpha_t(R, R') = 0$ otherwise.

For each loop c in $\mathcal{R}$ and each t ($1 \leq t \leq m-2$), we define $w_t(c) \in \mathbb{R}$ by

$$w_t(c) = \frac{1}{6} \sum_{k=1}^{\text{length}(c)} \alpha_t\big(c(k), c(k+1)\big).$$

An argument similar to Lemma 20 shows the following lemma:

Lemma 21. *For each pair of loops c, c' in $\mathcal{R}$ and each t ($1 \leq t \leq m-2$), if $c \simeq c'$, then $w_t(c) = w_t(c')$.*

For each t ($1 \leq t \leq m-2$) and each $\hat{R}^{l,t} \in \mathcal{L} \cap \hat{\mathcal{D}}_t$ (i.e., l is odd), the loop $\bar{c}_{\hat{R}^{l,t},t}$ of length 6 is defined the same as in Section 4. The definition of w_t immediately implies the following:

Lemma 22. *For each loop c in $\mathcal{R}$ and each t ($1 \leq t \leq m-2$), if* length$(c) = 6$, $w_t(c) = 1$, *and* $c(1) \in \hat{\mathcal{D}}_t$, *then* $c(1) \in \mathcal{L}$ *and* $c = \bar{c}_{c(1),t}$.

An argument similar to Lemma 12 together with Lemma 22 show the following lemma:

Lemma 23. *Suppose that c is a loop of length 6 in $\mathcal{R}$.*

(a) *If* $c(1) = \bar{R}$*, then either* $c \simeq (\bar{R})$*, or there exists* t *(*$1 \leq t \leq m-2$*) such that* $c = \bar{c}_{\bar{R},t}$ *or* $(\bar{c}_{\bar{R},t})^{-1}$.

(b) *If there exists* t *(*$1 \leq t \leq m-2$*) such that* $c(1) \in \hat{\mathcal{D}}_t$ *and* $c \simeq \bar{c}_{\bar{R},t}$*, then* $c(1) \in \mathcal{L}$ *and* $c = \bar{c}_{c(1),t}$.

Fix t ($1 \leq t \leq m-2$). Since $\bar{c}_{R,t}$ contains only linear orders for each $R \in \hat{D}_t \cap \mathcal{L}$, the social preference is a linear order whenever the assumptions in (a) or (b) in Lemma 22 holds. Thus, the same argument as in the proof of Lemma 13 derives

$$w_t\big(f(\gamma_1^{\bar{\boldsymbol{R}},t})\big) + \cdots + w_t\big(f(\gamma_n^{\bar{\boldsymbol{R}},t})\big) = 1.$$

We can show $f(\gamma_i^{\bar{\boldsymbol{R}},t}) \neq (\bar{c}_{\bar{R},t})^{-1}$ by the same argument as in the proof of Lemma 13 because $(\bar{c}_{\bar{R},t})^{-1}$ contains no preferences with ties. Since either $f(\gamma_i^{\bar{\boldsymbol{R}},t}) \simeq (\bar{R})$ or $f(\gamma_i^{\bar{\boldsymbol{R}},t}) = \bar{c}_{\bar{R},t}$ for each $i \in N$ by Lemma 23 (a), the above equality implies that there exists a unique agent $i_t \in N$ such that $f(\gamma_{i_t}^{\bar{\boldsymbol{R}},t}) = \bar{c}_{\bar{R},t}$ and $f(\gamma_j^{\bar{\boldsymbol{R}},t}) \simeq (\bar{R})$ for each $j \neq i_t$. By Lemmas 4 and 7, $f(\gamma_{i_t}^{\boldsymbol{R},t}) \simeq \bar{c}_{\bar{R},t}$ for each $\boldsymbol{R} \in \mathcal{L}^n$ such that $R_i \in \hat{\mathcal{D}}_t$. By Lemma 23 (b), if $f(\boldsymbol{R}) \in \hat{\mathcal{D}}_t$ then $f(\boldsymbol{R}) \in \mathcal{L}$ and $f(\gamma_{i_t}^{\boldsymbol{R},t}) = \bar{c}_{f(\boldsymbol{R}),t}$. Since $\bar{c}_{f(\boldsymbol{R}),t}$ contains no preferences with ties, the same argument as in the proof of Lemma 13 shows that i_t is independent of t.

Then the same argument as in the proof of Lemma 14 shows that there exists $i^* \in N$ such that for each $\boldsymbol{R}_{-i^*} \in \mathcal{L}^{n-1}$, $f(\bar{R}, \boldsymbol{R}_{-i^*}) = \bar{R}$. Finally, the argument the same as the one before Theorem 1 shows the desired result.

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