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A new criterion on manipulability of social choice functions

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Abstract

I propose a new axiom on the incentive for manipulating an outcome of a collective decision making. The axiom says that manipulation cannot occur through the preference relations adjacent to the sincere one. I discuss rationales for introducing the axiom and investigate the equivalence and the nonequivalence between the new axiom and strategy-proofness. I give examples of the nonequivalence, and find a sufficient condition for the equivalence. I show that the sufficient condition is satisfied by the universal domain and the set of single-peaked preferences. Accordingly, many existing results about strategy-proof social choice functions can be restated with the axiom proposed in this paper.

Keywords: adjacent manipulation, AM-proofness, single-peaked preferences, social choice function, strategy-proofness.

JEL classification: D71.

1 Introduction

The incentive to manipulate a social outcome is one of the most important topics in the theory of social choice. This paper proposes a new axiom on the robustness to manipulation of social choice functions (SCFs). I investigate the equivalence and the nonequivalence between my axiom and strategy-proofness over various domains and find a sufficient condition for the equivalence.

The axiom proposed in this paper says that an SCF is nonmanipulable through the preference relations adjacent to the sincere one. Two preference relations (formulated as linear orders) R and R' are *adjacent* if the only difference between them is one pair of consecutively ranked alternatives.[1] (For example, R and R' coincide except the ranks of x and y; in R, an alternative x is second ranked and y is third ranked; in R', y is second and x is third.) In this sense, adjacent preference relations are similar to each other.

My axiom will be called *AM-proofness*. ("AM" stands for "Adjacent Manipulation".) AM-proof SCFs are robust to manipulation by the adjacent preference relations. Because AM-proofness considers only adjacent preference relations as available options for misrepresentation, it is weaker than strategy-proofness. Rationales for AM-proofness are the following.

First, consider the *psychological cost* of misrepresenting the following two preference relations:

- a false preference relation close to the sincere one, and
- a false preference relation far from the sincere one.

Based on our intuition and experience, the cost of the latter is higher than that of the former. In other words, the further the preference relation is from the true one, the more reluctant an agent is to misreport the false preference relation.

[1] A formal definition of adjacent preference relations will be given in Section 2.

Besides the psychological cost, reporting a false preference relation different from the sincere one can cause immense damage to an agent even if the misrepresentation makes him better off in the social decision at hand. For example, consider a politician facing a collective decision making over an important political or economic issue. In many cases, his behavior (i.e., the reported preferences) is observable by the public. Then, reporting a false preference relation far from the sincere one has at least the following two negative effects. First, he loses his "real" supporters. Second, he is obliged to behave in line with the reported false preferences because lack of consistency lowers his reputation. The cost of the latter can be very high; his behavior and revelation of opinions are constrained to be consistent with the reported false preferences. As a result, he might unwillingly and unavoidably accomplish what he does not want at all.[2] It is intuitively clear that the further the reported false preferences is from the sincere one, the stronger these negative effects are. Therefore, for politicians, it would be good to reveal preferences relations "around" the sincere one.

From the above arguments, an SCF which is manipulable only through preference relations far from the sincere one (in other words, nonmanipulable through the similar preference relations) is more attractive in terms of the robustness to manipulation than an SCF which is manipulable through preference relations close to the sincere one. Therefore, AM-proofness serves as a useful criterion on manipulability of SCFs.[3]

[2]For example, consider an issue of gun control. Let a denote "no regulation", b denote "moderate gun control", and c denote "complete gun prohibition". Consider a politician with the following preferences: a is the most preferred, b is the second, and c is the third. Assume that (by some reasons) he misreports the converse at a committee on gun control: c is the most preferred, b is the second, and a is the third. Then, for consistency, at least for the following one or two years, he should behave as if he is a supporter for gun control everywhere: a committee, Congress, and small meetings. Meanwhile, he might establish gun control even though he *is* against any regulation.

[3]I would like to note that I do not claim that my axiom means a complete robustness to manipu-

However, why do we need to consider a new axiom on manipulability, AM-proofness, in spite of having strategy-proofness which is undoubtedly the most important axiom in this topic? There are mainly two reasons.

First, because strategy-proofness is a strong condition for nonmanipulability of SCFs, there is a large *conceptual* gap between strategy-proofness and non strategy-proofness. Therefore, more alternative axioms or measures of manipulability have been called for. Recent developments of the theory of manipulation fill this gap to some extent.[4] This research contributes to this line research by proposing an axiom weaker than strategy-proofness.

Second, in this paper, I examine the (non)equivalence between AM-proofness and strategy-proofness, and find a class of domains on which the new axiom and strategy-proofness are logically equivalent. Thus, further implications of strategy-proofness can be obtained.

In most of this paper, I concentrate on manipulation by adjacent preference relations and do not consider an "intermediate" notion between strategy-proofness and AM-proofness. I will formally show that once a sufficient condition for the equivalence between strategy-proofness and AM-proofness is established, then the condition is also sufficient for the equivalence between strategy-proofness and any "intermediate" notion of manipulability. (See Corollary 4.1.) Therefore, considering manipulation by adjacent preference relations can lead to a general result regarding the question of sufficiency for the equivalence which would be one of the most interesting and important problems.

Finally, I briefly review the related literature. The starting point of the literature would be the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite,

lation.

[4]See, for example, Campbell and Kelly (2009) and references therein. These works are mainly concerned with "measures" of manipulability, such as gains and losses due to manipulation and the number of preference profiles at which manipulation can occur.

1975).[5] Strategy-proofness has been the central axiom of the literature, but other axioms also have been considered. Le Breton and Zaporozhets (2009) and Barberà, Berga, and Moreno (2009) consider the problem of the equivalence between strategy-proofness and *coalitional* strategy-proofness. Coalitional strategy-proofness takes a possibility of coalition formation into account, and it is a stronger axiom than strategy-proofness. My work and theirs are in the same vein in that we explore the issue of the (non)equivalence of strategy-proofness and an alternative axiom.

While I weaken strategy-proofness by putting the restriction on the options of misrepresentation, Serizawa (2006) weakens coalitional strategy-proofness by putting the restriction on the size of coalitions and considers coalitions consisting of one or two agents because very large coalitions are practically difficult to form. (See also Schummer (2000).)

The paper is organized as follows. Section 2 gives basic notation and definitions. Section 3 gives examples of the nonequivalence between strategy-proofness and AM-proofness. Section 4 contains two Subsections; a sufficient condition and a necessary condition for the equivalence between strategy-proofness and AM-proofness are discussed in each Subsection. Section 5 discusses applications of my results.

2 Basic notation and definitions

Let $N = \{1, \ldots, n\}$ be a finite set of agents, let X be a finite set of alternatives with $|X| = m$[6], and let $\mathcal{L}$ denote the set of all linear orders[7] on X. An element of $\mathcal{L}$ will be called a *preference relation* and denoted R, R_i, and so on. Their strict parts are P and P_i. $R_N = (R_1, \ldots, R_n) \in \mathcal{L}^N$ is a preference profile, where the subscripts are indices of the agents. For each $R \in \mathcal{L}$ and for each $k \in \{1, \ldots, m\}$,

[5]Important works such as Black (1948) exist before the Gibbard-Satterthwaite theorem.

[6]For any set A, $|A|$ denotes its cardinality.

[7]A binary relation is called a linear order if it is complete, transitive, and antisymmetric.

let $r_k(R)$ denote the kth ranked alternative with respect to R.

Given $\mathcal{D} \subset \mathcal{L}$, a function f of $\mathcal{D}^N$ into X is called a *social choice function (SCF)* on $\mathcal{D}$, and $\mathcal{D}$ is called the *domain* of f.

Definition 2.1

- Given a preference relation R, two alternatives x and y are said to be *adjacent in* R if there exists $k \in \{1, \ldots, m-1\}$ such that $\{r_k(R), r_{k+1}(R)\} = \{x, y\}$.

- Two distinct preference relations R and R' are said to be *adjacent* if R' can be obtained by exchanging the ranks of one pair of adjacent alternatives in R; formally, there exist adjacent alternatives x, y in R such that $(R \setminus (x, y)) \cup (y, x) = R'$.[8] In this case, the set of adjacent alternatives $\{x, y\}$ is denoted by $A(R, R')$, and we say that y overtakes x from R to R'.

- Let $A(R)$ denote the set of all preference relations adjacent to R.

Example 2.1

Let $X = \{x, y, z\}$, and consider the two preference relations R and R' defined by $xRyRz$ and $yR'xR'z$. The only difference between R and R' is the ranks of x and y which are adjacent in R (and R'). Therefore, R and R' are adjacent, $A(R, R') = \{x, y\}$, $R' \in A(R)$, and y overtakes x from R to R'.

Definition 2.2

An SCF f on $\mathcal{D}$ is said to be *manipulable through an adjacent preference relation* if there exist $R_N \in \mathcal{D}^N$, $i \in N$, and $R'_i \in A(R) \cap \mathcal{D}$ such that $f(R'_i, R_{-i}) P_i f(R_N)$. The SCF f is *AM-proof* if it is not manipulable through an adjacent preference relation. Here, “AM” stands for “Adjacent Manipulation”.

Recall that in the definition of strategy-proofness[9], available options of misrepre-

[8] For notational simplicity, the braces around (x, y) and (y, x) are omitted.

[9] An SCF f is said to satisfy *strategy-proofness* if for any $R_N \in \mathcal{D}^N$, for any $i \in N$, and for any $R'_i \in \mathcal{D}$, $f(R_N) R_i f(R'_i, R_{-i})$ holds.

sentation are any preference relations in the domain. Therefore, under the universal domain $\mathcal{L}$, the number of available options of misrepresentation is $(m! - 1)$. On the other hand, because AM-proofness requires that f is robust to misrepresentation only through preference relations adjacent to the sincere one, the number of available options of misrepresentation in $\mathcal{L}$ is $(m - 1)$. Note that when $m = 10$, $(m! - 1) = 3628799$ and $(m - 1) = 9$, a great difference.

As we have seen, from the viewpoint of the number of preference relations which we have to check to establish a property, there exists a significant difference between strategy-proofness and AM-proofness on the universal domain. Generally speaking, the extent of the difference between strategy-proofness and AM-proofness depends on the domain under consideration. For example, if a domain consists of one pair of adjacent preference relations, then strategy-proofness and AM-proofness are equivalent. On the other hand, when any two preference relations in the domain $\mathcal{D}$ are not adjacent, then AM-proofness completely loses its power, and in this sense, the contrast between strategy-proofness and AM-proofness is sharper on $\mathcal{D}$ than on the universal domain. (See Section 3 for detailed arguments.)

In the following sections, I will demonstrate the (non)equivalence between strategy-proofness and AM-proofness.

3 The nonequivalence between strategy-proofness and AM-proofness

Clearly, on any domain $\mathcal{D}$, strategy-proofness implies AM-proofness. In this section, I will show that AM-proofness does not necessarily imply strategy-proofness through two kinds of domains.

Domains in which any two preference relations are not adjacent: First, I consider domains in which any two preference relations are not adjacent. On such a domain, every SCF is trivially AM-proof. On every domain containing two or more preference relations, an SCF violating strategy-proofness exists.[10] Therefore, the nonequivalence between AM-proofness and strategy-proofness arises.

Such a domain can be found in the literature. An example is minimal circular domains considered by Sato (2009). A domain $\mathcal{D}$ is said to be *circular* if the elements of X can be indexed $x_1, x_2, \ldots, x_m$ so that for each $k \in \{1, \ldots, m\}$, there exist two preferences R and R' in $\mathcal{D}$ such that

(i) $r_1(R) = x_k$, $r_2(R) = x_{k+1}$, $r_m(R) = x_{k-1}$,

(ii) $r_1(R') = x_k$, $r_2(R') = x_{k-1}$, and $r_m(R') = x_{k+1}$.

(Let $x_{m+1} = x_1$ and $x_0 = x_m$.)

In a circular domain $\mathcal{D}$, for each alternative x_k, there should be at least two preference relations R and R' satisfying the conditions (i) and (ii), respectively. Therefore, minimal (in terms of cardinality) circular domains contain exactly $2m$ preference relations. Sato (2009) proves that on any circular domain (especially, on minimal ones), every unanimous and nondictatorial SCF fails to satisfy strategy-proofness.

I can prove the following statement. (See Appendix for a proof.)

Proposition 3.1

When $m \geq 4$, in any minimal circular domains, any two preference relations are not adjacent.[11]

[10] Consider a domain $\mathcal{D}$ with $|\mathcal{D}| \geq 2$. Then, there exist distinct $x, y \in X$ with xRy and $yR'x$ for some $R, R' \in \mathcal{D}$. Let i be an element of N, let $R_i = R$, and let $R'_i = R'$. Let f be an SCF such that $f(R_i, R_{-i}) = y$ and $f(R'_i, R_{-i}) = x$ for some R_{-i}. Then, $f(R'_i, R_{-i}) P_i f(R_i, R_{-i})$ holds, which implies that f fails to satisfy strategy-proofness.

[11] When $m = 3$, the only circular domain is the universal domain. As I prove in Section 4, in the universal domain, every preference relation has an adjacent one.

Table 1:

R^1	R^2	R^3	R^4	R^5	R^6
x	y	y	y	x	x
y	x	x	x	y	y
v	v	v	z	z	v
w	w	z	v	v	z
z	z	w	w	w	w

Therefore, on the minimal circular domains, the nonequivalence between AM-proofness and strategy-proofness appears in a very sharp form; every unanimous and nondictatorial SCF satisfies AM-proofness but fails to satisfy strategy-proofness.

Domains in which any preference relation has an adjacent one: Next, I consider a domain in which every preference relation has an adjacent one.

Example 3.1

Let $N = \{1, 2\}$ and $X = \{x, y, v, w, z\}$. Consider the domain $\mathcal{D}$ consisting of the five preference relations from R^1 to R^5 in Table 1. (At this stage, ignore R^6.) Note that the pairs of adjacent preference relations in $\mathcal{D}$ are (R^1, R^2), (R^2, R^3), (R^3, R^4), (R^4, R^5). Therefore, for any preference relation $R \in \mathcal{D}$, there is a preference relation adjacent to R in the domain.

I prove that on $\mathcal{D}$, there exists an SCF satisfying AM-proofness but fails to satisfy strategy-proofness. Let f be the SCF defined by

$$f(R_1, R_2) = \begin{cases} x, & \text{if } R_1 = R^1, \\ y, & \text{otherwise.} \end{cases}$$

Thus, f takes x only if agent 1 reports R^1, and in the other cases, f takes y. I will show that the SCF f is AM-proof. To establish AM-proofness, it suffices to see the change of a social outcome induced by the (false) adjacent preference relations.

Under f, the change of a social outcome by an adjacent preference relation occurs only when agent 1 changes his report from R^1 to R^2 or from R^2 to R^1. In the first case, the social outcome changes from x to y, which is not beneficial with respect to R^1. In the second case, the social outcome changes from y to x, which is not beneficial with respect to R^2. Therefore, f is AM-proof.

However, f is not strategy-proof. A profitable misrepresentation by agent 1 is possible when his (true) preference relation is R^5. In this case, by falsely reporting R^1, the social outcome changes from y to x, which is a profitable change with respect to R^5.

The analysis in this section confirms the nonequivalence between strategy-proofness and AM-proofness. Therefore, some requirements on the domain are needed for the equivalence between the two conditions. To find such a requirement, the two kinds of the nonequivalence discussed here form a basis.

4 The equivalence between strategy-proofness and AM-proofness

In this section, I will present my main result: a sufficient condition for the equivalence between strategy-proofness and AM-proofness. A necessary condition for the equivalence is also discussed.

4.1 A sufficient condition

To describe a sufficient condition for the equivalence, I need several definitions.

Definition 4.1

A *path* from R to R' in $\mathcal{D} \subset \mathcal{L}$ is a sequence $(R^1 R^2 \dots R^\ell)$ in $\mathcal{D}$ such that

(i) R^h are all distinct ($h \in \{1, 2, \dots, \ell\}$),

(ii) $R = R^1$ and $R' = R^\ell$,

(iii) for each $h \in \{1, 2, \ldots, \ell - 1\}$, R^h is adjacent to R^{h+1}.

A path from R to R' is also a path from R' to R. Thus, when the direction is not important, we speak of path *between* R and R' in $\mathcal{D}$.

Definition 4.2

Two preference relations R and R' in $\mathcal{D} \subset \mathcal{L}$ are said to be *linked in* $\mathcal{D}$ if there exists a path between R and R' in $\mathcal{D}$.

Definition 4.3

A domain $\mathcal{D} \subset \mathcal{L}$ is said to be *connected* if any two preference relations in $\mathcal{D}$ are linked.

Assumption 4.1

For any R and R' in $\mathcal{D} \subset \mathcal{L}$, if they are linked, then there exists a path $(R^1 \ldots R^\ell)$ between R and R' in $\mathcal{D}$ such that $A(R^h, R^{h+1}) \neq A(R^k, R^{k+1})$ for all distinct $h, k \in \{1, \ldots, \ell - 1\}$. Such a path will be called a *non-twice overtaking path*. In a non-twice overtaking path, for any $x, y \in X$, once x overtakes y in some step in the sequence, say R^h to R^{h+1}, then y never overtakes x in the rest of the sequence.

Example 3.1 (continued)

Let $\mathcal{D}$ be the domain consisting the five preference relations from R^1 to R^5 in Table 1. Then, this $\mathcal{D}$ is connected, while it violates Assumption 4.1; from R^1 to R^5, there is a unique path in $\mathcal{D}$, that is, $(R^1 R^2 R^3 R^4 R^5)$. Because y overtakes x in the step from R^1 to R^2, and x overtakes y in the step from R^4 to R^5, $A(R^1, R^2) = A(R^4, R^5) = \{x, y\}$. Therefore, $\mathcal{D}$ does not satisfy Assumption 4.1.

Remark 4.1

Given a sequence $(R^1 R^2 \ldots R^\ell)$ in $\mathcal{D} \subset \mathcal{L}$ satisfying the second and the third conditions of Definition 4.1, if $A(R^h, R^{h+1}) \neq A(R^k, R^{k+1})$ for all distinct R^h and R^k in the sequence (i.e., it is non-twice overtaking), then the first condition of Definition 4.1 is satisfied. To see this, I prove that for any R^h in the sequence, $R^h \neq R^k$

for all $k \geq h+1$. From R^h to R^{h+1}, the ranks of one pair of adjacent alternatives are exchanged. Let x and y denote these alternatives (i.e., $A(R^h, R^{h+1}) = \{x, y\}$), and assume that xR^hy and $yR^{h+1}x$. Because the sequence is non-twice overtaking, for any R^k with $k \geq h+1$, $A(R^k, R^{k+1}) \neq \{x, y\}$. In other words, x never overtakes y after R^h. Therefore, yR^kx holds for any $k \geq h+1$, and hence $R^h \neq R^k$ for any $k \geq h+1$. Because h was arbitrary, this implies that any two preference relations in the sequence are distinct from each other. Therefore, to prove that a sequence under consideration is a non-twice overtaking path, it suffices to check the second and the third conditions of Definition 4.1 and the non-twice overtaking property.

The following is a main theorem of this paper; it finds a sufficient condition for the equivalence between strategy-proofness and AM-proofness.

Theorem 4.1

If a domain $\mathcal{D}$ is connected and satisfies Assumption 4.1, then AM-proofness implies strategy-proofness. In other words, any SCF satisfying AM-proofness satisfies strategy-proofness.

Lemma 4.1

Let f be any AM-proof SCF on a domain $\mathcal{D}$. Then, for any $R_N \in \mathcal{D}^N$, for any $R'_i \in A(R_i) \cap \mathcal{D}$ such that $[f(R_N)R_iy \Rightarrow f(R_N)R'_iy \;\; \forall y \in X]$, we have $f(R'_i, R_{-i}) = f(R_N)$.

Proof of Lemma 4.1. Suppose $f(R'_i, R_{-i}) \neq f(R_N)$. If $f(R_N)P_if(R'_i, R_{-i})$, then $f(R_N)P'_if(R'_i, R_{-i})$, which is a contradiction to AM-proofness. (Note that $R_i \in A(R'_i) \cap \mathcal{D}$.) If $f(R'_i, R_{-i})P_if(R_N)$, then it is also a contradiction to AM-proofness of f. ■

Proof of Theorem 4.1. Let f be an SCF on $\mathcal{D}$ satisfying AM-proofness. Let R_N be any preference profile and let i be any agent. Let R'_i be any preference in $\mathcal{D}$. Because $\mathcal{D}$ is connected and satisfies Assumption 4.1, there exists a non-twice overtaking path $(R^1R^2 \dots R^\ell)$ in $\mathcal{D}$ from R_i to R'_i with $R_i = R^1$ and $R'_i = R^\ell$. Our goal is to prove $f(R_i, R_{-i})R_if(R'_i, R_{-i})$, i.e., $f(R'_i, R_{-i})$ belongs to the lower counter set (in a weak sense) of $f(R_i, R_{-i})$ with respect to R_i. Let $L = \{y \in X \mid f(R_i, R_{-i})R_iy\}$ (the set of lower contour set of $f(R_N)$ with respect to R_i).

STEP 1: By Lemma 4.1, in each step from R^h to R^{h+1} $(h \in \{1, \dots, \ell - 1\})$ in the sequence $(R^1R^2 \dots R^\ell)$, the social outcome can change only when the alternative right below $f(R^h, R_{-i})$ overtakes $f(R^h, R_{-i})$. (In the other cases, the lower contour set of $f(R^h, R_{-i})$ expands in a weak sense, and hence the social outcome does not change.)

STEP 2: I prove that for each $h = 1, \dots, \ell-1$, if $f(R^h, R_{-i}) \in L$, then $f(R^{h+1}, R_{-i}) \in L$. By Step 1, it suffices to consider the case where from R^h to R^{h+1}, the alternative x right below $f(R^h, R_{-i})$ overtakes $f(R^h, R_{-i})$.
CASE 1: $x \in L$. In this case, $f(R^{h+1}, R_{-i})$ is either $f(R^h, R_{-i})$ or x. In either case, $f(R^{h+1}, R_{-i}) \in L$.
CASE 2: $xP_if(R_i, R_{-i})$. Because $f(R^h, R_{-i}) \in L$, at the starting point $R^1(= R_i)$, x was ranked higher than $f(R^h, R_{-i})$. Because x is now just below $f(R^h, R_{-i})$, $f(R^h, R_{-i})$ overtook x in some earlier stage in the sequence $(R^1 \dots R^h)$. Remember the sequence under consideration is a non-twice overtaking path, which implies that x never overtakes $f(R^h, R_{-i})$ in the sequence. Therefore, this case cannot occur.

STEP 3: Trivially, $f(R^1, R_{-i}) = f(R_i, R_{-i}) \in L$. Then, by the induction on h and Step 2, it follows that $f(R'_i, R_{-i}) = f(R^\ell, R_{-i}) \in L$, i.e., $f(R_i, R_{-i})R_if(R'_i, R_{-i})$, and we complete the proof. ■

Example 3.1 (continued)

Let $\mathcal{D}'$ be the preference relations from R^1 to R^6 in Table 1. Then, it can be seen that $\mathcal{D}'$ is connected and satisfies the non-twice overtaking property; from R^1 to R^5, $(R^1 R^6 R^5)$ is a non-twice overtaking path; from R^6 to a non-adjacent preference relation (R^2 and R^4), there exists a non-twice overtaking path ($(R^6 R^3 R^2)$ and (R^6, R^3, R^4), respectively).

By Theorem 4.1, strategy-proofness and AM-proofness are equivalent on $\mathcal{D}'$, while not on $\mathcal{D} = \{R^1, R^2, R^3, R^4, R^5\}$ (see page 9). This is an example where the existence or admissibility of one preference relation (in this case, R^6) is crucial for the equivalence.

Theorem 4.1 shows the equivalence between strategy-proofness and AM-proofness on a class of domains satisfying the assumptions in the theorem. From Theorem 4.1, we readily have the following corollary which extends the equivalence result to any "intermediate" notions of the robustness to manipulation.

Corollary 4.1

Let f be an SCF on $\mathcal{D}$. Under the assumptions of Theorem 4.1, the following statements are equivalent.

(i) f is strategy-proof.

(ii) For any $R_N \in \mathcal{D}^N$ and for any $i \in N$,

$$f(R_N) R_i f(R_i', R_{-i}), \quad \forall R_i' \in M_i(R_N),$$

where M_i is a correspondence of $\mathcal{D}^N$ into $\mathcal{D}$ satisfying $M_i(R_N) \supset (A(R_i) \cap \mathcal{D})$ for all $R_N \in \mathcal{D}^N$. (Remember that $A(R_i)$ is the set of all preference relations adjacent to R_i.)

Proof. (i)⇒(ii) is trivial. I prove (ii)⇒(i). The second statement implies that $f(R_N)R_if(R'_i, R_{-i})$ for all R_N, for all $i \in N$, and for all $R'_i \in (A(R_i) \cap \mathcal{D})$. Thus, f is AM-proof. By Theorem 4.1, f is strategy-proof. ■

This corollary shows the equivalence between strategy-proofness and any condition which is "between" strategy-proofness and AM-proofness. Thus, even if you consider that AM-proofness is unacceptably extreme and more options for misrepresentation should be taken into account, we can still have the same equivalence result with your formulation of the robustness to manipulation.

4.2 A necessary condition

In Theorem 4.1, connectedness of a domain is a part of the sufficient condition for the equivalence between strategy-proofness and AM-proofness. In the following proposition, I prove that connectedness is also a necessary condition for the equivalence.

Proposition 4.1

If strategy-proofness and AM-proofness are equivalent on a domain $\mathcal{D}$, then $\mathcal{D}$ is connected.

Proof.[12] Assume that strategy-proofness and AM-proofness are equivalent on a domain $\mathcal{D}$. Suppose to the contrary that $\mathcal{D}$ is not connected. Consider a(n) (undirected) graph $G = (V, E)$, where $V = \mathcal{D}$ and $\{R, R'\} \in E$ if R and R' are adjacent. Because $\mathcal{D}$ is not connected, the graph G is not connected. Hence, G has at least two components: let $G^1 = (\mathcal{D}^1, E^1), G^2 = (\mathcal{D}^2, E^2), \ldots, G^k = (\mathcal{D}^k, E^k)$ denote the components $(k \geq 2)$. Let h^* be any element of $\{1, \ldots, k\}$ and it is fixed in the rest of this proof. Note that $\mathcal{D} = \bigcup_{h=1}^{k} \mathcal{D}^h$.

[12] In this proof, I use basic concepts in graph theory. The reader is referred to a standard textbook on graph theory, such as Diestel (2005), for the definitions.

If there exists $x^* \in X$ such that $r_1(R) = x^*$ for all $R \in \mathcal{D}$, then define f by

$$f(R_N) = \begin{cases} r_2(R_1) & \text{if } R_1 \in \bigcup_{h \neq h^*} \mathcal{D}^h, \\ x^* & \text{if } R_1 \in \mathcal{D}^{h^*}. \end{cases}$$

The SCF f is AM-proof but not strategy-proof. Therefore, AM-proofness and strategy-proofness are not equivalent, which is a contradiction.

Next, I consider the case where there exist R and R' in $\mathcal{D}$ such that $r_1(R) = x$ and $r_1(R') = y$ for some distinct $x, y \in X$. Then, define f by

$$f(R_N) = \begin{cases} r_1(R_1) & \text{if } R_1 \in \bigcup_{h \neq h^*} \mathcal{D}^h, \\ r_1(R_2) & \text{if } R_1 \in \mathcal{D}^{h^*}. \end{cases}$$

The SCF is AM-proof, but not strategy-proof. Therefore, AM-proofness and strategy-proofness are not equivalent, which is a contradiction. ■

In the previous subsection, I noted that the sufficient condition for the equivalence between strategy-proofness and AM-proofness is also sufficient for the equivalence between strategy-proofness and an "intermediate" notion of robustness to manipulation, which is an important property because AM-proofness would not be the only candidate for notions weaker than strategy-proofness. (See Corollary 4.1.) However, a necessary condition does not have such an applicability to "intermediate" notions. In this sense, a necessary condition is not as telling as a sufficient condition.

Nevertheless, Proposition 4.1 clarifies an important point; even when the equivalence holds on a domain $\mathcal{D}$, it does not necessarily imply the equivalence on superdomains of $\mathcal{D}$. This can be seen by observing that adding preference relations to $\mathcal{D}$ can lead to *non*connectedness. This makes a contrast to some other properties of domains.[13]

[13] For example, Sanver (2007) proves that when every alternative is top ranked by some preference relation in a dictatorial domain $\mathcal{D}$, then any superdomain of $\mathcal{D}$ inherits the dictatorial property from

5 Applications

In this section, I will give examples of domains satisfying the assumptions of Theorem 4.1.

There are several important domains in the theory of social choice. The universal domain and classes of single-peaked preferences would be the most important ones.

5.1 The universal domain

Arrow's theorem (Arrow, 1963), Sen's theorem (Sen, 1970), and the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) are all proved on the universal domain. Because these impossibility theorems lay the foundation of social choice theory as a modern discipline, the analysis on the universal domain should be conducted whenever we have a new axiom on social choice rules.

Because all possible preference relations are in $\mathcal{L}$, it seems natural to expect that $\mathcal{L}$ is connected, i.e., there is a path between any two preference relations. However, the question of the existence of a non-twice overtaking path is not straightforward.

Proposition 5.1

The domain $\mathcal{L}$ satisfies the assumptions of Theorem 4.1.

Proof. Let R and R' be any preference relations in $\mathcal{L}$. I construct a non-twice overtaking path from R to R'.

STEP 1: Compare $r_1(R)$ and $R_1(R')$. If $r_1(R) = r_1(R')$, then proceed to the next step. If $r_1(R) \neq r_1(R')$, then raise $r_1(R')$ one position at a time to the top in R.

...

STEP K: Let R^* denote the resulting preference relation from Step $(k-1)$. Compare $r_k(R^*)$ and $r_k(R')$, the kth ranked alternatives in R^* and R', respectively. If

$\mathcal{D}$.

they are identical, then proceed to the next step. If $r_k(R^*) \neq r_k(R')$, then raise $r_k(R')$ one position at a time to the kth position in R^*.

This process of transformation terminates at Step m. I claim that all preference relations appear in this process (including the ones which appear in the process of raising $r_k(R^*)$ one position at a time in each step) form a non-twice overtaking path from R to R'.

[Adjacency]:

Because every change in each preference relation occurs by exchanging the ranks of adjacent alternatives, every preference relation is adjacent to the "next" preference relation. (Remember that $r_k(R^*)$ is raised *one position at a time* in Step k ($k \in \{1, \ldots, m\}$).)

[Non-twice overtaking property]:

If R^* is the resulting preference relation from Step k, then the top-k alternatives (from $r_1(R^*)$ to $r_k(R^*)$) in R^* are never affected in the following steps. In this sense, R is adjusted to R' from the top to the bottom. Let x and y be any alternatives. Assume that x overtakes y in Step k ($k \in \{1, \ldots, m\}$). This means that x is ranked higher than y in R' and that $x = r_k(R')$. Because the top-k ranks are never affected in the following steps, for any $k' > k$, y does not overtake x in Step k'. Thus, the sequence under consideration is non-twice overtaking.

Therefore, a sequence from R to R' satisfies the conditions (ii) and (iii) of Definition 4.1 and has a non-twice overtaking property. By Remark 4.1, the preference relations in the above process form a non-twice overtaking path from R to R'. ■

Remark 5.1

Let R and R' be any elements of $\mathcal{L}$ and let $(R^1 R^2 \ldots R^\ell)$ be the non-twice overtaking path from R to R' constructed in the proof of Proposition 5.1. Then, for any $x, y \in X$, [xRy and $xR'y$] implies [$xR^h y$ for all $h \in \{1, \ldots, \ell\}$].

Proof. Suppose that $yR^h x$ for some $h \in \{1, \ldots, \ell\}$. Because xRy, there exists

$h' \in \{1, \ldots, h\}$ such that y overtakes x from $R^{h'}$ to $R^{h'+1}$. However, because $xR'y$, there exists $h'' \in \{h'+1, \ldots, \ell\}$ such that x overtakes y from $R^{h''}$ to $R^{h''+1}$, which is a contradiction to the non-twice overtaking property. ■

5.2 Single-peaked preference relations

Another significant domain is a set of single-peaked preferences (Black, 1948). A collection of preference relations $\mathcal{D} \subset \mathcal{L}$ is said to be a set of *single-peaked preference relations* if the elements of X can be indexed $x_1, x_2, \ldots, x_m$ so that for any $R \in \mathcal{D}$ with $r_1(R) = x_k$ and for any $k', k'' \in \{1, \ldots, m\}$,

(i) $k < k' < k'' \Rightarrow x_{k'} R x_{k''}$, and

(ii) $k'' < k' < k \Rightarrow x_{k'} R x_{k''}$.

In the rest of this section, the alternatives are indexed as $x_1, \ldots, x_m$, and whenever I speak of single-peaked preference relations, it should be understood that they are single-peaked with respect to the indices of the alternatives.

What makes single-peaked preferences significant is its applicability and evasion from the impossibilities. See, for example, Black (1948), Moulin (1980, 1988).

The first question which I deal with is whether *any* collection of single-peaked preference relations satisfies the assumptions of Theorem 4.1. The following example gives a negative answer.

Example 5.1

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, let $n = 5$, and let $\mathcal{D}$ denote the set of all single-peaked preference relations (with respect to the indices of the alternatives) R such that $r_1(R) = \{x_1, x_3, x_5\}$. Note that there is only one preference relation in $\mathcal{D}$ which ranks x_1 at the top and that there is only one preference relation in $\mathcal{D}$ which ranks x_5 at the top. Consider the plurality rule with the tie-breaking rule $x_1 >$

$x_3 > x_5$, which means that a tie between x_1 and x_3 is broken by choosing x_1, and so on. I claim that this plurality rule is not strategy-proof while it is AM-proof on $\mathcal{D}$.

First, to see that the plurality rule is not strategy-proof, consider the following situation;

- $r_1(R_i) = x_1$ for $i = 1, 2$,
- $r_1(R_3) = x_3$ with $x_5 R_3 x_1$.
- $r_1(R_i) = x_5$ for $i = 4, 5$.

Then, under the sincere preference relations, the social outcome is x_1. Then, a profitable misrepresentation is possible for agent 3 by reporting a false single-peaked preference relation R'_3 such that $r_1(R'_3) = x_5$. (In this case, a social outcome changes from x_1 to x_5, which is a beneficial change with respect to R_3.) Thus, the plurality rule is not strategy-proof on $\mathcal{D}$.

Next, to see that the plurality rule is AM-proof on $\mathcal{D}$, observe that any adjacent preference relations in $\mathcal{D}$ should have a common top. (Actually, for a preference relation $R \in \mathcal{D}$ with $r_1(R) \in \{x_1, x_5\}$, $A(R) \cap \mathcal{D} = \emptyset$.) Because the plurality rule depends on only what are top ranked, the social outcome never changes by reporting an adjacent preference relation in $\mathcal{D}$. Therefore, the plurality rule is AM-proof on $\mathcal{D}$.

The above example shows that additional restrictions are called for the equivalence between strategy-proofness and AM-proofness. I consider the cases where the social choice rule designer knows that the admissible preference relations are single-peaked, but has no further information. In other words, the domain is the set of all possible single-peaked preference relations.

In the following, let $\mathcal{S}$ denote the collection of all single-peaked preference relations. On $\mathcal{S}$, the existence of a non-twice overtaking path is a more complicated issue than on $\mathcal{L}$. Nevertheless, we can prove the following proposition.

Proposition 5.2

The domain $\mathcal{S}$ satisfies the assumptions of Theorem 4.1.

Proof. Let R and R' be any elements of $\mathcal{S}$. As in the proof of Proposition 5.1, I construct a non-twice overtaking path from R to R' in $\mathcal{S}$.

STEP 1: Compare $r_1(R)$ and $r_1(R')$. If they are identical, then proceed to Step 3. If $r_1(R) \neq r_1(R')$, then proceed to Step 2.

STEP 2: There are two possible cases: $r_1(R)$ is in the right of $r_1(R')$, or $r_1(R)$ is in the left of $r_1(R')$. *Here, I consider only the former case.* (In the latter case, a non-twice overtaking path from R to R' can be constructed by a symmetric argument.) Let $x_k = r_1(R)$ and let $x_{k'} = r_1(R')$. Because x_k is in the right of $x_{k'}$, $k' < k$.

SUBSTEP 2.1: Raise x_{k-1} to the top one position at a time in R.

...

SUBSTEP 2.h ($k' \leq k - h \leq k - 1$): Let R^* be the preference relation resulting from Substep 2.$(h-1)$. Raise x_{k-h} to the top one position at a time in R^*.

In this process, the peak moves from the right to the left; from x_k to $x_{k'}$. Let $\tilde{R}$ denote the preference relation resulting from Substep 2.$(k - k')$.

[Single-peakedness]:

First, because the only operation in each substep is raising the alternatives just on the left of the peak to the top, every preference relations appearing in this whole process are single-peaked.

[Adjacency]:

Next, because every change of preference relations occurs by exchanging the ranks of one pair of adjacent alternatives, every preference relation in this process is adjacent to the next preference relation.

[Non-twice overtaking property]:

Third, in the sequence of preference relations from R to $\tilde{R}$, x_{k-h} overtakes an alternative only in the process of raising x_{k-h} to the top (at Substep 2.$(k-h)$). In this case, the former peak is x_{k-h+1} and hence the alternatives which x_{k-h} overtakes are all on the right of x_{k-h}. The alternatives on the right of x_{k-h} never raise their position in the following substeps. Especially, they do not overtake x_{k-h}. Therefore, the sequence from R to $\tilde{R}$ is a non-twice overtaking path.

STEP 3: Transform $\tilde{R}$ to R' according to the transformation process in the proof of Proposition 5.1. Note that the most preferred alternatives are always $x_{k'}$ throughout the transformation process.

[Single-peakedness]:

To prove that all preference relations in the process are single-peaked, it suffices to show that the relative positions *within* each side of the "mountain" do not change. For that purpose, I consider any two alternatives x_ℓ and $x_{\ell'}$ in the left of the peak $x_{k'}$ (i.e., $\ell, \ell' < k'$). Without loss generality, assume $\ell < \ell'$. Because $\tilde{R}$ and R' are single-peaked with a common peak $x_{k'}$, $x_{\ell'} \tilde{R} x_\ell$ and $x_{\ell'} R' x_\ell$. By Remark 5.1, this relative ranking between x_ℓ and $x_{\ell'}$ never changes throughout the transformation process from $\tilde{R}$ to R'. Because x_ℓ and $x_{\ell'}$ were arbitrary in the left side of $x_{k'}$, the relative rankings *within* the left side of the peak do not change throughout the process. By the symmetric argument, I can prove that relative rankings *within* the right side of the peak also never change throughout the process. Therefore, all preference relations appear in the process from $\tilde{R}$ to R' are single-peaked.

[Non-twice overtaking property]:

Note that because the sequence constructed in the proof of Proposition 5.1 is a non-twice overtaking path, the sequence from $\tilde{R}$ to R' in this Step 3 is also a non-twice overtaking path.

STEP 4: I prove that a sequence of preferences from R to R' constructed in Step 2 and Step 3 is a non-twice overtaking path in $\mathcal{S}$. The sequence is in $\mathcal{S}$ because all preference relations appear in Step 2 and Step 3 are single-peaked. Thus, it suffices to show that the sequence is non-twice overtaking.

[Non-twice overtaking property]:

By Step 2 and Step 3, the sequences *within* each step are non-twice overtaking. Therefore, it suffices to show that when an alternative x_ℓ overtakes $x_{\ell'}$ in Step 2, then $x_{\ell'}$ does not overtake x_ℓ in Step 3. Note that x_ℓ raises its position and overtakes other alternatives including $x_{\ell'}$ in Step 2 only when the peak $x_{\ell+1}$ is replaced by x_ℓ. This implies that the alternatives which x_ℓ overtakes in Step 2 are all in the right of x_ℓ. Therefore, $\ell' > \ell$. Moreover, because x_ℓ becomes the peak at some substep in Step 2, the peak of $\tilde{R}$ is either x_ℓ or an alternative in the left of x_ℓ. Thus, either $x_\ell = x_{k'}$ or x_ℓ and $x_{\ell'}$ are on the same side of the "mountain" under $\tilde{R}$. In the former case, because x_ℓ is the peak throughout Step 3, $x_{\ell'}$ never overtakes x_ℓ. In the latter case, because the relative positions *within* each side of the 'mountain" do not change in Step 3, $x_{\ell'}$ does not overtake x_ℓ throughout Step 3. In any case, $x_{\ell'}$ does not overtake x_ℓ.

Therefore, the sequence constructed in Step 2 and Step 3 is a non-twice overtaking path from R to R' in $\mathcal{S}$. ■

Appendix A proof of Proposition 3.1

Let $\mathcal{D}$ be any minimal circular domain and let R be any element of $\mathcal{D}$. Let x_k denote $r_1(R)$.

STEP 1: First, let R' be any preference relation in $\mathcal{D}$ such that $r_1(R') = x_{k'}$ with

$k' \neq k$. If R and R' are adjacent, then $r_2(R) = x_{k'}$. This implies either $x_{k'} = x_{k-1}$ or $x_{k'} = x_{k+1}$. In the former case, $r_m(R) = x_{k+1}$ and $r_m(R') = x_{k-2}$. Because $m \geq 4$, $x_{k+1} \neq x_{k-2}$. In the latter case, $r_m(R) = x_{k-1}$ and $r_m(R') = x_{k+2}$. Because $m \geq 4$, $x_{k-1} \neq x_{k+2}$. Thus, by exchanging the top two alternatives in R, we cannot have R'. Therefore, R and R' are not adjacent.

STEP 2: The only possible candidate for an adjacent preference relation to R is $\tilde{R}$ such that $r_1(\tilde{R}) = x_k$. However, the second ranked alternative and the bottom ranked alternative are different between R and $\tilde{R}$. Because $m \geq 4$, the second ranked alternative and the bottom ranked alternative cannot be adjacent. Therefore, R and $\tilde{R}$ are not adjacent.

References

Arrow, Kenneth J. (1963) Social choice and individual values, 2nd edn. John Wiley, New York.

Barberà, Salvador; Berga, Dolors; Moreno, Bernardo (2009) Individual versus group strategy-proofness: when do they coincide? Barcelona Economics Working Paper Series WP372.

Black, Duncan (1948) On the rationale of group decision-making. Journal of Political Economy 56, 23–34.

Campbell, Donald E.; Kelly, Jerry S. (2009) Gains from manipulating social choice rules. Economic Theory 40, 349–371.

Diestel, Reinhard (2005) Graph theory. Springer, Berlin.

Gibbard, Allan (1973) Manipulation of voting scheme: A general result. Econometrica 41, 587–601.

Le Breton, Michel; Zaporozhets, Vera (2009) On the equivalence of coalitional and individual strategy-proofness properties. Social Choice and Welfare 33, 287–309.

Moulin, Hervé (1980) On strategy-proofness and single peakedness. Public Choice 35, 437–455.

Moulin, Hervé (1988) Axioms of cooperative decision making. Cambridge University Press, Cambridge.

Sanver, M. Remzi (2007) A characterization of superdictatorial domains for strategy-proof social choice functions. Mathematical Social Sciences 54, 257–260.

Satterthwaite, Mark A. (1975) Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10, 187–217.

Sato, Shin (2009) Circular domains. mimeo.

Schummer, James (2000) Manipulation through bribes. Journal of Economic Theory 91, 180–198.

Sen, Amartya K. (1970) The impossibility of a Paretian liberal. Journal of Political Economy 72, 152–157.

Serizawa, Shigehiro (2006) Pairwise strategy-proofness and self-enforcing manipulation. Social Choice and Welfare 26, 305–331.