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On strategy-proof social choice under categorization

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Abstract

I consider the social choice problems such that (i) the set of alternatives can be categorized into two groups based on a prominent and objective feature and (ii) the agents have strict preferences over the alternatives. I introduce a notion of linear dichotomous domains to describe a class of plausible preference profiles in these social choice problems. A main finding is a characterization of the form of all strategy-proof social choice functions on any linear dichotomous domain.

Keywords: categorization, decomposability, linear dichotomous domain, strategy-proofness.

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1 Introduction

In this paper, I consider the social choice problems such that (i) the set of alternatives can be categorized into two groups based on some prominent and objective feature and (ii) the agents have strict preferences over the alternatives. My main result is a characterization of the form of strategy-proof social choice functions (SCFs) in these social choice problems.

First, I give a representative example of social choice problems which I consider in this paper. Consider an election under the two-party system. Assume that every candidate belongs to either Party L or Party R and that the voters have preferences over the candidates. The social choice problem is to choose one candidate as a winner. Assume that political standpoints of two parties, Party L and Party R, are clearly distinct from each other. Because the candidates in the same party should share views about basic aspects of controversial political issues, when a political standpoint is the most important feature of each candidate (this would the case in most elections with some exceptions[1]), plausible preferences would be one of the followings:

(i) Any candidate in Party L is preferred to any candidate in Party R, or

(ii) Any candidate in Party R is preferred to any candidate in Party L.

Moreover, each voter presumably has strict preferences over the candidates in each party. This example is one of the social choice problems described in the first sentence of this paper; The voters have strict preferences, and the set of candidates is categorized based on political parties, which is a prominent and objective feature of each candidate.

We can find other examples easily. When a group of agents considers a plan for the winter vacation and the candidates are Michigan, Bahamas, and Jamaica,

[1]For example, candidates' gender and race might be important factors in some elections.

then the set of alternatives is naturally categorized into two groups, {Michigan} and {Bahamas, Jamaica}. (This example is based on Saari (2001).) Moreover, in the choice of a trade policy by politicians or officials, the available options might be categorized into the following two groups: one is a set of policies favoring free trade and the other is a set of policies favoring protective trade.

These examples suggest that categorization or classification of the alternatives often occurs in social choice problems. Therefore, social choice under categorization is an important subject of research. This is not a new topic, and hence I do not claim novelty of considering such situations. However, to the best of my knowledge, the structure of strategy-proof SCFs in such a situation has never been investigated. Therefore, I explore this issue.

My main result is a decomposability theorem; An SCF is strategy-proof if and only if the SCF can be decomposed into the following three strategy-proof "local" SCFs;

(i) One SCF makes a choice between the two groups, and

(ii) the other two SCFs choose an alternative from each group.

In Section 2, I give basic notation and definitions. In Section 3, a main result and its applications are presented. In Section 4, I review some related literature. Proofs are collected in Section 5.

2 Basic notation and definitions

Let $N = \{1, \ldots, n\}$ be a finite set of *agents*.

First, I give notation and definitions with respect to an arbitrary set of alternatives A. Let $\mathcal{L}(A)$ denote the set of all linear orders[2] on A. Linear orders will be denoted by R, R_i, $\succsim$, $\succsim_i$, and so on. Their strict part will be denoted by P, P_i, $\succ$,

[2]A binary relation is called a linear order if it is complete, transitive, and antisymmetric.

and $\succ_i$. For each $R \in \mathcal{L}(A)$, let $r_k(R)$ denote the kth ranked alternative in A with respect to R. A function F of $\mathbb{D} \subset \mathcal{L}(A)^N$ into A is called a *social choice function (SCF) on* $\mathbb{D}$, and $\mathbb{D}$ is called the domain of F. An SCF F on $\mathbb{D} \subset \mathcal{L}(A)^N$ is

- *strategy-proof* if for any $i \in N$, for any $R_N \in \mathbb{D}$, and for any $R'_i \in \mathcal{L}(A)$ such that $(R'_i, R_{-i}) \in \mathbb{D}$, the relation $F(R_N) R_i F(R'_i, R_{-i})$ holds.

- *unanimous* if for any $a \in A$ and for any $R_N \in \mathbb{D}$ such that $r_1(R_i) = a$ for all $i \in N$, $F(R_N) = a$.

- *essential*[3] if for any $i \in N$, there exist R_N and $\tilde{R}_N \in \mathbb{D}$ such that $R_j = \tilde{R}_j$ for all $j \in N \setminus \{i\}$ and $F(R_N) \neq F(\tilde{R}_N)$.

When F is strategy-proof, then for any agent, it is always an optimal strategy to reveal his sincere preferences. When F is unanimous, then any alternative which is top ranked by all agents is the social outcome. In the definition of an essential SCF, the only difference between R_N and $\tilde{R}_N$ in the above definition is agent i's preference. Thus, when agent i changes his preference from R_i to $\tilde{R}_i$, then the social outcome also changes. In this sense, when F is essential, every agent can be "pivotal" at some preference profile, and moreover, because every agent can affect a social outcome, there is no dictator.

In the rest of this paper, let X denote a finite set of *alternatives*, and consider social choice problems of choosing a social outcome from X. As the domains of SCFs choosing a social outcome from X, I restrict my attention to $\mathbb{D} \subset \mathcal{L}(X)^N$ such that $\mathbb{D} = \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{n \text{ times}}$ for some $\mathcal{D} \subset \mathcal{L}(X)$. This assumption is equivalent to the following two statements;

(i) admissibility of preferences of one agent does not depend on other agents' preferences, and

[3] To the best of my knowledge, this condition is first considered by Blair and Muller (1983).

(ii) each agent has a common set of admissible preferences.

In the following definition, I give a core concept of this paper. Remember that, in this paper, I consider the social choice problems such that (i) the set of alternatives can be categorized into two groups by some prominent and objective feature and (ii) the agents have strict preferences over the alternatives. The domain defined in the following definition is intended to describe a class of plausible preference profiles in such situations.

Definition 2.1

A set $\mathcal{D} \subset \mathcal{L}(X)$ is said to be *linear dichotomous* with respect to a partition $\{X_1, X_2\}$ of X if for any $R \in \mathcal{D}$, either

(i) $x_1 P x_2$ for all $(x_1, x_2) \in X_1 \times X_2$, or

(ii) $x_2 P x_1$ for all $(x_1, x_2) \in X_1 \times X_2$ holds.

When $\mathcal{D}$ is linear dichotomous, a preference R in $\mathcal{D}$ is called a *linear dichotomous preference* and the set $\mathbb{D} = \mathcal{D} \times \cdots \times \mathcal{D}$ is called a *linear dichotomous domain*.

When $\mathbb{D}$ is a linear dichotomous domain, then the set of alternatives X can be divided into two groups X_1 and X_2 such that either (i) any alternative in X_1 is preferred to any alternative in X_2 or (ii) any alternative in X_2 is preferred to any alternative in X_1.

I deal with linear dichotomous domains because in many social choice problems, the set of admissible preference profiles is linear dichotomous. We have already seen several examples in the introduction. But, as long as the set of alternatives can be classified into two groups by some prominent and objective feature, then the resulting preferences would be linear dichotomous.

In the rest of this paper, a partition $\mathcal{X} = \{X_1, X_2\}$ of X is fixed, and $\mathbb{D} = \mathcal{D} \times \cdots \times \mathcal{D} \subset \mathcal{L}(X)^N$ always refers to a linear dichotomous domain with respect to $\mathcal{X}$.

I introduce notation necessary for later discussion. Given $R \in \mathcal{D}$, let $R|_{X_1}$ denote the restriction of R to X_1. Formally, $R|_{X_1} = R \cap (X_1 \times X_1)$. Let $\mathcal{D}|_{X_1}$ be the collection of such restrictions, that is, $\{R|_{X_1} \mid R \in \mathcal{D}\}$. Let $R|_{X_2}$ denote the restriction of R to X_2, and let $\mathcal{D}|_{X_2} = \{R|_{X_2} \mid R \in \mathcal{D}\}$. Let $R|_{\mathcal{X}}$ denote the linear order on $\mathcal{X}$ induced by R. Let $\mathcal{D}|_{\mathcal{X}} = \{R|_{\mathcal{X}} \mid R \in \mathcal{D}\}$ and let $\mathbb{D}|_{\mathcal{X}} = \underbrace{\mathcal{D}|_{\mathcal{X}} \times \cdots \times \mathcal{D}|_{\mathcal{X}}}_{n \text{ times}}$.

Given an SCF f on $\mathbb{D}$, let

- $\mathbb{D}_{X_1} = \{R_N|_{X_1} \mid R_N \in \mathbb{D} \text{ and } f(R_N) \in X_1\}$, where $R_N|_{X_1} = (R_1|_{X_1}, \dots, R_n|_{X_1})$.
- $\mathbb{D}_{X_2} = \{R_N|_{X_2} \mid R_N \in \mathbb{D} \text{ and } f(R_N) \in X_2\}$.

The set $\mathbb{D}_{X_1}$ is the collection of the restrictions of $R_N \in \mathbb{D}$ to X_1 such that the social outcome at R_N is in X_1. Note that the relations $\mathbb{D}_{X_1} \subset \mathcal{D}|_{X_1} \times \cdots \times \mathcal{D}_{X_1}$ and $\mathbb{D}_{X_2} \subset \mathcal{D}|_{X_2} \times \cdots \times \mathcal{D}_{X_2}$ always hold, but the converse (set inclusion) relations do *not* necessary hold. Also, one of $\mathbb{D}_{X_1}$ and $\mathbb{D}_{X_2}$ can be empty, but they cannot be simultaneously empty. (If both $\mathbb{D}_{X_1}$ and $\mathbb{D}_{X_2}$ are empty, then f never takes a value in $X_1 \cup X_2$, which is a contradiction to the fact that X_1 and X_2 form a partition of X.)

3 Results

In this section, I give a main theorem and related results on strategy-proof SCFs on a linear dichotomous domain $\mathbb{D}$.

3.1 The decomposability theorem

The following theorem is a main result of this paper, which I call *the decomposability theorem*.

Theorem 3.1 (Decomposability Theorem)

Let f be an SCF on $\mathbb{D}$. The following statements are equivalent:

(i) The SCF f is strategy-proof.

(ii) The SCF f can be written as follows; For each $R_N \in \mathbb{D}$,

$$f(R_N) = \begin{cases} g^1(R_N|_{X_1}), & \text{if } h(R_N|_{\mathcal{X}}) = X_1, \\ g^2(R_N|_{X_2}), & \text{if } h(R_N|_{\mathcal{X}}) = X_2, \end{cases} \tag{3.1}$$

where g^1 is a strategy-proof SCF on $\mathbb{D}_{X_1}$ provided that $\mathbb{D}_{X_1} \neq \emptyset$, g^2 is a strategy-proof SCF on $\mathbb{D}_{X_2}$ provided that $\mathbb{D}_{X_2} \neq \emptyset$, and h is a strategy-proof SCF on $\mathbb{D}|_{\mathcal{X}}$. If $\mathbb{D}_{X_1} = \emptyset$, then (3.1) consists of only h and g^2, and if $\mathbb{D}_{X_2} = \emptyset$, then (3.1) consists of only h and g^1.

There are three functions in the right hand side of the equation (3.1). The function h makes a choice between X_1 and X_2 based on the induced preference profile over $\mathcal{X}$. The functions g^1 and g^2 choose from X_1 and X_2 based on the restricted preferences $R_N|_{X_1}$ and $R_N|_{X_2}$, respectively. Statement (ii) of the theorem implies that the process of choosing a social outcome from X according to a strategy-proof SCF f can be decomposed into these three "local" SCFs if there is no chance of profitable misrepresentation.

I give two remarks on the decomposability theorem.

First, this decomposability result can be derived only from strategy-proofness and linear dichotomous domains. No other conditions are assumed. (Remember that most results on strategy-proof social choice come with additional conditions. Examples of such conditions are properties of SCFs, richness of domains, and cardinality requirement of the set of alternatives.)

Second, this theorem is neither the possibility theorem nor the impossibility theorem. This theorem just characterizes the *form* of strategy-proof SCFs on a linear dichotomous domain $\mathbb{D}$. However, as the following analysis shows, we can have some direct implications from our decomposability theorem.

3.2 A deviation from the impossibility in the spirit of the Gibbard-Satterthwaite theorem

In this subsection, I construct a strategy-proof, unanimous, and essential SCF on linear dichotomous domains satisfying the following assumption.

Assumption 3.1

In $\mathcal{D}$, there exist R and $\tilde{R}$ such that $r_1(R) \in X_1$ and $r_1(\tilde{R}) \in X_2$.

This is a weak requirement for "richness" of $\mathbb{D} = \mathcal{D} \times \cdots \times \mathcal{D}$.

Proposition 3.1

Under Assumption 3.1, there exists a strategy-proof, unanimous, and essential SCF on $\mathbb{D}$.

When an SCF is essential, *every* agent can affect a social outcome. Hence there is no dictator under the SCF, i.e., it is nondictatorial. Therefore, under a very weak condition Assumption 3.1, we can deviate from the impossibility in the spirit of the Gibbard-Satterthwaite theorem[4] (Gibbard, 1973; Satterthwaite, 1975).

However, it is too hasty to conclude that we can always construct a "desirable" SCF.

In the following, when I speak of h, g^1, and g^2, they should be understood to be those in Theorem 3.1 associated with a strategy-proof SCF f under consideration.

A domain is said to be *dictatorial* if any strategy-proof and unanimous SCF on the domain is dictatorial.[5]

Proposition 3.2

Assume that an SCF f on $\mathbb{D}$ is strategy-proof and unanimous, then

[4] The Gibbard-Satterthwaite theorem states that there is no strategy-proof and unanimous SCF on $\mathcal{L}(X)$ provided that $|X| \geq 3$.

[5] See, for example, Aswal, Chatterji, and Sen (2003) for results on dictatorial domains.

(i) h, g^1, and g^2 are all unanimous. (If $\mathbb{D}_{X_1} = \emptyset$, then g^1 is absent from the representation, and if $\mathbb{D}_{X_2} = \emptyset$, then g^2 is absent from the representation.)

(ii) g^1 is a dictatorship if $\mathbb{D}_{X_1}$ is a dictatorial domain and g^2 is a dictatorship if $\mathbb{D}_{X_2}$ is a dictatorial domain.

Note that the second statement of the proposition is a direct consequence of Theorem 3.1 and the first statement of Proposition 3.2. According to this proposition, although it is true that we can construct a strategy-proof, unanimous, and essential SCF on $\mathbb{D}$ under a weak assumption (Proposition 3.1), when $\mathbb{D}_{X_1}$ or $\mathbb{D}_{X_2}$ is a dictatorial domain, then there always exists a "local" dictator.

Remarks on Proposition 3.2

(i) When $\mathbb{D}_{X_1}$ and $\mathbb{D}_{X_2}$ are dictatorial, then g^1 and g^2 are dictatorship. In this case, it is possible that dictators under g^1 and g^2 are distinct.

(ii) The first statement of the proposition says that unanimity of f is carried over to the "smaller" SCFs: h, g^1, and g^2. However, some standard property, such as neutrality, of f cannot be carried over to h, g^1, and g^2.[6]

4 Related literature

In this section, I review related literature.

First, Inada (1964) considers a class of domains containing special cases of linear dichotomous domains.[7] In his Theorem 4, Inada (1964) considers domains

[6]As a simple example, consider the case where $X = \{x, y, z, v, w\}$, $X_1 = \{x, y, z\}$, and $X_2 = \{v, w\}$. Let $\mathcal{D} = \{yzxvw, zyxvw, zyxwv, vwxyz, wvxzy\}$. This is a linear dichotomous domain. The point is that x is always at the same position. Then, let f be the constant function which always takes the value x. Clearly, f is strategy-proof and neutral. However, it can be seen that the SCF h associated with f is not neutral.

[7]Some domains considered by Inada (1964) are not linear dichotomous just because Inada (1964) deals with weak orders.

of a social welfare function such that "any set of a given number of alternatives greater than two can be separated into two groups of alternatives". For example, if the set of alternatives X is separated into two groups X_1 and X_2 such that $|X_1| > 2$, then the set X_1 should be further separable into two groups in the sense of Definition 2.1 with a slight difference. (Inada (1964) allows indifferences within each group.)

In terms of domains, Sakai and Shimoji (2006) would be most closely related to my research. They consider a set of weak orders $\mathcal{D}_i$ such that there exists a partition G_i and B_i of X such that $\mathcal{D}_i$ is the set of all weak orders on X such that any alternative in G_i is preferred to any alternative in B_i. (G_i is a set of "good" alternatives and B_i is a set of "bad" alternatives.) An agent having such a set of admissible preferences is called a *dichotomist* by Sakai and Shimoji (2006). In my analysis, when $\mathcal{D}_i$ is linear dichotomous with respect to a partition X_1 and X_2 of X, it is possible that in some preference in $\mathcal{D}_i$, X_1 is set of good alternatives and X_2 is a set of bad alternatives while in some other preference in $\mathcal{D}_i$, X_2 is a set of good alternatives and X_1 is a set of bad alternatives. Moreover, many linear dichotomous sets of linear orders can be associated with a partition X_1 and X_2 while, in Sakai and Shimoji (2006), it is uniquely determined once a partition G_i and B_i of X is given.

The concept of linear dichotomous preferences is a variant of that of dichotomous preferences. To the best of my knowledge, dichotomous preferences are first considered by Inada (1964), and the term "dichotomous preferences" is first used by Brams and Fishburn (1978).[8]

My decomposability theorem is reminiscent of the result by Le Breton and Sen (1999). They give the decomposability theorem in a different context where the set of alternatives can be written as the Cartesian product of "component" sets.

[8]See, for example, Bogomolnaia, Moulin, and Stong (2005) and Vorsatz (2008) for recent studies on dichotomous preferences.

They prove that when every admissible preference relation is separable, then any SCF can be decomposed into strategy-proof "marginal" SCFs choosing an outcome from each component set.

I discussed the existence of local dictators in X_1 and X_2 in the last paragraph of the previous section. In the context of *social welfare functions*, a similar result is given by Fishburn (1976). Fishburn considers a problem of aggregation of individual preferences into an *incomplete* social preference relation. To identify the pairs of alternatives between which a social comparison is to be made, Fishburn considers an undirected graph (X, E); a social comparison between x and y is to be made iff $\{x, y\} \in E$. The set of individual preferences is the collection of all asymmetric binary relations $\succ$ on X such that $x \succ y$ or $y \succ x$ iff $\{x, y\} \in E$ for all $\{x, y\} \in E$, and do not have a cycle on any circuit in (X, E). Fishburn proves that under standard conditions, a dictator exists *within* each block and different blocks can have different dictators. Moreover, each edge does not necessarily have a dictator. (See, for example, Diestel (2005) for the definitions of each concept of graph theory.) There are many differences between Fishburn's approach and mine. In addition to several clear differences, the most significant difference is that Fishburn's main concern is (in)completeness of social orderings and he essentially deals with the universal domain whereas my main concern is domain restriction.[9]

5 Proofs

In this section, $\mathbb{D}$ always refers to a linear dichotomous domain with respect to a partition $\mathcal{X} = \{X_1, X_2\}$. First, I give a lemma which will be used repeatedly in

[9]For example, in Fishburn's formulation, if a preference relation on X is admissible, then its converse relation (or the dual relation) is necessarily admissible. (If the original relation is asymmetric, the converse is also asymmetric. Also, if the original relation does not have a cycle on any circuit, then its converse has the same property.) However, it is not the case in my model.

	R_i	$\tilde{R}_i$	R_i	$\tilde{R}_i$	
Best	X_2	X_1	X_1	X_2	$R_i\|_{\mathcal{X}} = \tilde{R}_i\|_{\mathcal{X}}$
Worst	X_1	X_2	X_2	X_1	
Agents	$1, \ldots, n''$		$n''+1, \ldots, n'$		$n'+1, \ldots, n$

Table 1: The preference profiles R_N and $\tilde{R}_N$ in the proof of Lemma 5.1

later arguments.

Lemma 5.1

Let f be any strategy-proof SCF on $\mathbb{D}$ Then, for any $t \in \{1, 2\}$, for any two preference profiles R_N and $\tilde{R}_N$ in $\mathbb{D}$ such that $R_N|_{X_t} = \tilde{R}_N|_{X_t}$, $f(R_N) \in X_t$, and $f(\tilde{R}_N) \in X_t$, we have $f(R_N) = f(\tilde{R}_N)$.

Proof. I give a proof for the case $t = 1$. Without loss of generality, let $\{1, \ldots, n'\}$ be the subset of N defined by $\{i \in N \mid R_i|_{\mathcal{X}} \neq \tilde{R}_i|_{\mathcal{X}}\}$ and let $\{n'+1, \ldots, n\}$ be the subset of N defined by $\{i \in N \mid R_i|_{\mathcal{X}} = \tilde{R}_i|_{\mathcal{X}}\}$. Assume that for any $i \in \{1, \ldots, n''\}$ $(n'' \leq n')$, $X_2 R_i|_{\mathcal{X}} X_1$ holds while for any $i \in \{n''+1, \ldots, n'\}$, $X_1 R_i|_{\mathcal{X}} X_2$ holds. Because $R_i|_{\mathcal{X}} \neq \tilde{R}_i|_{\mathcal{X}}$ for all $i \in \{1, \ldots, n'', \ldots, n'\}$, $X_1 \tilde{R}_i|_{\mathcal{X}} X_2$ for any $i \in \{1, \ldots, n''\}$ and $X_2 \tilde{R}_i|_{\mathcal{X}} X_1$ for any $i \in \{n''+1, \ldots, n'\}$. The situation under consideration is summarized in Table 1. Note that R_i and $\tilde{R}_i$ are preferences on X. For example, the second column of Table 1 represents that under R_i, the alternatives in X_1 is preferred to the alternatives in X_2.

Consider the successive change of preferences from R_N to $\tilde{R}_N$. First, consider agent 1's change of his preferences from R_1 to $\tilde{R}_1$. Because $f(R_N) \in X_1$, by strategy-proofness of f, the alternatives in X_2 cannot be chosen at the new profile $(\tilde{R}_1, R_{-1})$. Therefore, $f(\tilde{R}_1, R_{-1}) \in X_1$. Because of $R_i|_{X_1} = \tilde{R}_i|_{X_1}$ (remember the assumption $R_N|_{X_1} = \tilde{R}_N|_{X_1}$) and strategy-proofness of f, $f(\tilde{R}_1, R_{-1}) = f(R_N)$. By the same reasoning, after the agents from 1 to n'' change their preferences from R_i to $\tilde{R}_i$, the value of f is still $f(R_N)$.

Given the change of preferences of agents 1 through n'', next, consider that the agents from $n''+1$ to n' change their preferences successively. First, consider agent $n''+1$'s change of his preferences from $R_{n''+1}$ to $\tilde{R}_{n''+1}$. If the value of f is in X_2 after this change, then by strategy-proofness of h, the value of f never comes back to X_1 through the process of successive change of preferences. This is a contradiction to the assumption $f(\tilde{R}_N) \in X_1$. Therefore, f should choose an alternative in X_1. Moreover, because $R_{n''+1}|_{X_1} = \tilde{R}_{n''+1}|_{X_1}$, the choice from X_1 cannot change. (If it does change, then it leads to a contradiction to strategy-proofness of f.) Therefore, the value of f is still $f(R_N)$. By the same reasoning, after the agents from $n''+1$ to n' change their preferences from R_i to $\tilde{R}_i$, the value of f is still $f(R_N)$.

What I have proved so far is that after the change of preferences of agents 1 through n', the social outcome is still $f(R_N)$. Finally, consider the change of preferences of the agents $n'+1$ through n. First, consider agent $n'+1$'s change of his preferences from $R_{n'+1}$ to $\tilde{R}_{n'+1}$. By strategy-proofness of h (and f), f chooses an alternative in X_1. Because $R_{n'+1}|_{X_1} = \tilde{R}_{n'+1}|_{X_1}$, the choice from X_1 cannot change, and hence $f(R_N)$ is still chosen. By the same reasoning, after all agents change their preferences from R_i to $\tilde{R}_i$, the value of f is still $f(R_N)$, that is, $f(\tilde{R}_N) = f(R_N)$.

By the symmetric argument, we can give a proof for the case $t = 2$.

5.1 A proof of Theorem 3.1

(i) $\Rightarrow$ (ii): Let f be any strategy-proof SCF on $\mathbb{D}$.

STEP 1: For each $R_N \in \mathbb{D}$, define

$$h(R_N|\mathcal{X}) = \begin{cases} X_1, & \text{if } f(R_N) \in X_1, \\ X_2, & \text{if } f(R_N) \in X_2. \end{cases}$$

I prove that this h is a well-defined function on $\mathbb{D}|_{\mathcal{X}}$; formally, for any R_N and $\tilde{R}_N$ in $\mathbb{D}$ such that $R_N|_{\mathcal{X}} = \tilde{R}_N|_{\mathcal{X}}$, the relation $h(R_N|_{\mathcal{X}}) = h(\tilde{R}_N|_{\mathcal{X}})$ holds. The assumption $R_N|_{\mathcal{X}} = \tilde{R}_N|_{\mathcal{X}}$ implies that the differences between R_N and $\tilde{R}_N$ exist only "within" X_1 and X_2. Therefore, if $f(R_N) \in X_s$ and $f(\tilde{R}_N) \in X_t$ with distinct $s, t \in \{1, 2\}$, then it is a contradiction to strategy-proofness of f.

STEP 2: I prove that the function h is strategy-proof. If $\mathcal{D}|_{\mathcal{X}}$ is a singleton (and hence $\mathbb{D}|_{\mathcal{X}}$ is also a singleton), then h is a constant function and it is trivially strategy-proof. Consider the case where $\mathcal{D}|_{\mathcal{X}}$ consists of the two linear orders on $\mathcal{X}$, that is, $\mathcal{D}|_{\mathcal{X}} = \mathcal{L}(\mathcal{X})$. Let $\succsim^1$ denote the linear order such that $X_1 \succ^1 X_2$, where $\succ^1$ is the strict part of $\succsim^1$, and let $\succsim^2$ denote the linear order such that $X_2 \succ^2 X_1$.

Suppose that h is not strategy-proof. First, consider the case where for some $i \in N$, $h(\succsim_i^1, \succsim_{-i}) \succ_i^2 h(\succsim_i^2, \succsim_{-i})$ holds for some profile without agent i's preference $\succsim_{-i}$. That is, agent i can manipulate by changing his preference $\succsim_i^2$ to $\succsim_i^1$. This is possible only if $h(\succsim_i^1, \succsim_{-i}) = X_2$ and $h(\succsim_i^2, \succsim_{-i}) = X_1$. Then, f chooses an alternative in X_2 when agent i reports a (false) preference such that the alternatives in X_1 are preferred to the alternatives in X_2 while f chooses an alternative in X_1 when he reports a (true) preference such that the alternatives in X_2 are preferred to the alternatives in X_1. This is a contradiction to strategy-proofness of f.

Similarly, the case of $h(\succsim_i^2, \succsim_{-i}) \succ_i^1 h(\succsim_i^1, \succsim_{-i})$ for some $i \in N$ leads to a contradiction.

STEP 3: Consider the case where h is a constant function. Without loss of generality, assume that $h(\succsim_N) = X_1$ for all $\succsim_N \in \mathbb{D}|_{\mathcal{X}}$. Then, $\mathbb{D}_{X_2} = \emptyset$, and hence it suffices to construct g^1.

For each $R_N \in \mathbb{D}$, define $g^1(R_N|_{X_1}) = f(R_N)$. Frist, I prove that this g^1

is a well-defined function on $\mathbb{D}_{X_1}$. Let R_N and $\tilde{R}_N$ be any preference profiles in $\mathbb{D}$ such that $R_N|_{X_1} = \tilde{R}_N|_{X_1}$, and I will prove that $f(R_N) = f(\tilde{R}_N)$. Consider the successive change of preferences from R_N to $\tilde{R}_N$. Because h is a constant function, the value of f is always in X_1. Remember that all agents do not change preferences within X_1 (i.e., $R_N|_{X_1} = \tilde{R}_N|_{X_1}$). Thus, if the value of f changes at some stage in the successive change from R_N to $\tilde{R}_N$, then it is a contradiction to strategy-proofness of f. Therefore, $f(R_N) = f(\tilde{R}_N)$, and hence g_1 is a well-defined function on $\mathbb{D}_{X_1}$.

Next, I prove that g^1 is strategy-proof. Let $\succsim_N$ be any element of $\mathbb{D}_{X_1}$, let i be any agent in N, and let $\succsim'_i$ be any element of $\mathcal{L}(X_1)$ such that $(\succsim'_i, \succsim_{-i}) \in \mathbb{D}_{X_1}$. I will show that $g^1(\succsim_N) \succsim_i g^1(\succsim'_i, \succsim_{-i})$. Let R_N be an element of $\mathbb{D}$ such that $R_N|_{X_1} = \succsim_N$ and let R'_i be an element of $\mathcal{D}$ such that $R'_i|_{X_1} = \succsim'_i$. Because $f(R_N), f(R'_i, R_{-i}) \in X_1$, $f(R_N) = g^1(\succsim_N)$ and $f(R'_i, R_{-i}) = g^1(\succsim'_i, \succsim_{-i}$). Because f is strategy-proof, $f(R_N) R_i f(R'_i, R_{-i})$ holds. Therefore, $g^1(\succsim_N$)$R_i g^1(\succsim'_i, \succsim_{-i})$, which implies that $g^1(\succsim_N) \succsim_i g^1(\succsim'_i, \succsim_{-i})$.

STEP 4: Consider the case where h takes both X_1 and X_2, that is, $f(R_N) \in X_1$ for some $R_N \in \mathbb{D}$ and $f(\tilde{R}_N) \in X_2$ for some $\tilde{R}_N \in \mathbb{D}$. In this case, for each $R_N \in \mathbb{D}$ such that $f(R_N) \in X_1$, define $g^1(R_N|_{X_1}) = f(R_N)$ and for each $R_N \in \mathbb{D}$ such that $f(R_N) \in X_2$, define $g^2(R_N|_{X_2}) = f(R_N)$.

I prove that g^1 is a well-defined function on $\mathbb{D}_{X_1}$. Let R_N and $\tilde{R}_N$ be any preference profiles in $\mathbb{D}$ such that $f(R_N) \in X_1$, $f(\tilde{R}_N) \in X_1$, and $R_N|_{X_1} = \tilde{R}_N|_{X_1}$. The goal is to prove the relation $g^1(R_N|_{X_1}) = g^1(\tilde{R}_N|_{X_1})$. By Lemma 5.1, $f(R_N) = f(\tilde{R}_N)$ holds. Because $f(R_N) \in X_1$ and $f(\tilde{R}_N) \in X_1$, by the definition of g^1, $g^1(R_N|_{X_1}) = f(R_N) = f(\tilde{R}_N) = g^1(\tilde{R}_N|_{X_1})$.

By the symmetric argument, I can prove that g^2 is a well-defined function on $\mathbb{D}_{X_2}$.

STEP 5: I prove that g^1 is strategy-proof. Let i be any agent, let $\succsim_N$ be any element of $\mathbb{D}_{X_1}$ and let $\tilde{\succsim}_i$ be any element of $\mathcal{L}(X_1)$ such that $(\tilde{\succsim}_i, \succsim_{-i}) \in \mathbb{D}_{X_1}$. The goal is to prove the relation $g^1(\succsim_N) \succsim_i g^1(\tilde{\succsim}_i, \succsim_{-i})$. Because $\succsim_N \in \mathbb{D}_{X_1}$ and $(\tilde{\succsim}_i, \succsim_{-i}) \in \mathbb{D}_{X_1}$, there exist $R_N \in \mathbb{D}$ and $\tilde{R}_N \in \mathbb{D}$ such that $R_N|_{X_1} = \succsim_N$, $\tilde{R}_N|_{X_1} = (\tilde{\succsim}_i, \succsim_{-i})$, $f(R_N) \in X_1$, and $f(\tilde{R}_N) \in X_1$. Without loss of generality, assume that $i = n$. Then, $R_{-n}|_{X_1} = \succsim_{-n} = \tilde{R}_{-n}|_{X_1}$. There are three cases to consider:

(i) $R_n|_{\mathcal{X}} = \tilde{R}_n|_{\mathcal{X}}$.

(ii) $X_1 R_n|_{\mathcal{X}} X_2$ and $X_2 \tilde{R}_n|_{\mathcal{X}} X_1$.

(iii) $X_2 R_n|_{\mathcal{X}} X_1$ and $X_1 \tilde{R}_n|_{\mathcal{X}} X_2$.

Case (i): In this case, because $R_n|_{\mathcal{X}} = \tilde{R}_n|_{\mathcal{X}}$ and f is strategy-proof, $f(R_n, \tilde{R}_{-n}) \in X_1$ holds. (Remember that $f(\tilde{R}_N) \in X_1$.) Moreover, $R_N|_{X_1} = (R_n, \tilde{R}_{-n})|_{X_1}$ and $f(R_N) \in X_1$ hold. Then, by Lemma 5.1, $f(R_N) = f(R_n, \tilde{R}_{-n})$ holds. By strategy-proofness of f, $f(R_n, \tilde{R}_{-n}) R_n f(\tilde{R}_N)$, which is equivalent to $f(R_N) R_n f(\tilde{R}_N)$. Because $f(R_N) \in X_1$, $f(\tilde{R}_N) \in X_1$, $R_N|_{X_1} = \succsim_N$, $\tilde{R}_N|_{X_1} = (\tilde{\succsim}_n, \succsim_{-n})$, and $R_n|_{X_1} = \succsim_n$, we have $g^1(\succsim_N) \succsim_n g^1(\tilde{\succsim}_n, \succsim_{-n})$.

Case (ii): In this case, $f(R_n, \tilde{R}_{-n}) \in X_1$ holds. (If $f(R_n, \tilde{R}_{-n}) \in X_2$, then by strategy-proofness of f, $f(\tilde{R}_n, \tilde{R}_{-n}) \in X_2$ holds, which is a contradiction to the assumption $f(\tilde{R}_N) \in X_1$.) Then, by the same reasoning as in Case (i), we can conclude that $g^1(\succsim_N) \succsim_n g^1(\tilde{\succsim}_n, \succsim_{-n})$ holds.

Case (iii): In this case, $f(\tilde{R}_n, R_{-n}) \in X_1$ holds. (If $f(\tilde{R}_n, R_{-n}) \in X_2$, then by strategy-proofness of f, $f(R_n, R_{-n}) \in X_2$ holds, which is a contradiction.) Moreover, $\tilde{R}_N|_{X_1} = (\tilde{R}_n, R_{-n})|_{X_1}$ and $f(\tilde{R}_N) \in X_1$ hold. Then, by Lemma 5.1, we have $f(\tilde{R}_N) = f(\tilde{R}_n, R_{-n})$. By strategy-proofness, $f(R_N) R_n f(\tilde{R}_n, R_{-n})$, which is equivalent to $f(R_N) R_n f(\tilde{R}_N)$. Because $f(R_N) \in X_1$, $f(\tilde{R}_N) \in X_1$, $R_N|_{X_1} = \succsim_N$, $\tilde{R}_N|_{X_1} = (\tilde{\succsim}_n, \succsim_{-n})$ and $R_n|_{X_1} = \succsim_n$, we have $g^1(\succsim_N) \succsim_n$

$g^1(\tilde{\succsim}_n, \succsim_{-n})$.

Therefore, in any case, g^1 is strategy-proof on $\mathbb{D}_{X_1}$. By the symmetric argument, it can be seen that g^2 is strategy-proof.

(ii) ⇒ (i): Assume that the SCF f can be written as in the statement (ii) of Theorem 3.1. Suppose to the contrary that f is not strategy-proof. Then, there exist $i \in N$, $R_N \in \mathbb{D}$, and $\tilde{R}_i \in \mathcal{D}$ such that $f(\tilde{R}_i, R_{-i}) P_i f(R_N)$ holds. There are two cases to consider.

Case 1: $f(\tilde{R}_i, R_{-i}) \in X_s$ and $f(R_N) \in X_t$ for distinct $s, t \in \{1, 2\}$. Remember that g^1 chooses from X_1 and g^2 chooses from X_2, and which of g^1 and g^2 to use to make a social decision is determined by the function h (see equation (3.1)). Then, $f(\tilde{R}_i, R_{-i}) \in X_s$ implies that $h(\tilde{R}_i|_{\mathcal{X}}, R_{-i}|_{\mathcal{X}})$ should be X_s, and $f(R_N) \in X_t$ implies that $h(R_N|_{\mathcal{X}})$ should be X_t. Because $X_s R_i|_{\mathcal{X}} X_t$, $h(\tilde{R}_i|_{\mathcal{X}}, R_{-i}|_{\mathcal{X}}) P_i|_{\mathcal{X}} h(R_N|_{\mathcal{X}})$ holds, which is a contradiction to strategy-proofness of h.

Case 2: $f(\tilde{R}_i, R_{-i}) \in X_t$ and $f(R_N) \in X_t$ for some $t \in \{1, 2\}$. In this case, $f(\tilde{R}_i, R_{-i}) = g^t(\tilde{R}_i|_{X_t}, R_{-i}|_{X_t})$ and $f(R_N) = g^t(R_N|_{X_t})$ hold. Then, $f(\tilde{R}_i, R_{-i}) P_i f(R_N)$ implies $g^t(\tilde{R}_i|_{X_t}, R_{-i}|_{X_t}) P_i|_{X_t} g^t(R_N|_{X_t})$ holds, which is a contradiction to strategy-proofness of g^t.

5.2 A proof of Proposition 3.1

In the following, I show that when $\mathbb{D}$ satisfies Assumption 3.1, then there exists a strategy-proof, unanimous, and essential SCF on $\mathbb{D}$.

Such an SCF can be constructed as follows; First, choose any two distinct agents i_1 and i_2 from N. For each preference profile $R_N \in \mathbb{D}$,

STEP 1: choose either X_1 or X_2 by the simple majority rule at $R_N|_{\mathcal{X}}$. (If n is odd, then the simple majority rule is an SCF on $\mathcal{L}(\mathcal{X})$. If n is even, then some tie-breaking rule is need for the simple majority rule to be an SCF on $\mathcal{L}(\mathcal{X})$. I use the tie-breaking rule choosing X_1 when X_1 and X_2 have the same number of supports.)

STEP 2: When X_t wins at Step 1, then choose the most preferred alternative in X_t with respect to $R_{i_t}|_{X_t}$.

Let f denote this SCF on $\mathbb{D}$. Let h denote the simple majority rule in Step 1 and let g^t denote the SCF on $\mathbb{D}_{X_t}$ in Step 2.

[Strategy-proof:] Because h, g^1, and g^2 are all strategy-proof, by Theorem 3.1, the SCF f is strategy-proof.

[Unanimous:] When every agent put $x \in X$ at the top, then the group to which x belongs wins at Step 1 and either i_1 or i_2 chooses x in Step 2. Therefore, f is unanimous.

[Essential:] By Assumption 3.1, $\mathcal{D}|_{\mathcal{X}} = \mathcal{L}(\mathcal{X})$, which is implicitly used in Step 1. Let i be any agent.

Assume that n is even. Let $n' = n/2$. At $R_{-i} \in \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{(n-1)\ \text{times}}$, assume that $X_1 R_j|_{\mathcal{X}} X_2$ for $n' - 1$ agents and $X_2 R_j|_{\mathcal{X}} X_1$ for the other n' agents. (Note that "n'" in this context is just an integer and not a name of an agent.) Then, when agent i prefers X_1 to X_2, then X_1 is chosen in Step 1 while when agent i prefers X_2 to X_1, then X_2 is chosen in Step 1. Clearly, a final social outcome is affected by agent i's preference.

Assume that n is odd. Then, let $n' = (n-1)/2$. At $R_{-i} \in \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{(n-1)\ \text{times}}$, assume that $X_1 R_j|_{\mathcal{X}} X_2$ for n' agents and $X_2 R_j|_{\mathcal{X}} X_1$ for the other n' agents. Again, agent i is decisive in the choice between X_1 and X_2.

Therefore, f is essential.

5.3 A proof of Proposition 3.2

Because the second statement of the proposition is a direct consequence of the first statement, I prove only the first statement.

STEP 1: I prove that h is unanimous. First, assume that $\mathcal{D}|_{\mathcal{X}}$ is a singleton. In this case, $\mathbb{D}|_{\mathcal{X}}$ is also a singleton. Let $\succsim_N$ be the element of $\mathbb{D}|_{\mathcal{X}}$. (Note that $\succsim_i = \succsim_j$ for all $i, j \in N$.) Assume that $X_s \succ_i X_t$ $(s, t \in \{1, 2\})$ for all $i \in N$. Let R be an element of $\mathcal{D}$. Because $\mathcal{D}|_{\mathcal{X}}$ is a singleton and $\succsim$ is the element of $\mathcal{D}|_{\mathcal{X}}$, $R|_{\mathcal{X}} = \succsim$. Let $R_N = (R, \ldots, R)$. By unanimity of f, $f(R_N) = r_1(R) \in X_s$. Therefore, $h(\succsim_N) = X_s$, and hence h is unanimous.

Next, assume that $\mathcal{D}|_{\mathcal{X}} = \mathcal{L}(\mathcal{X})$. Let X_t be either X_1 or X_2 and let X_s be the element of $\mathcal{X} \setminus \{X_t\}$. Let $\succsim_N$ be the element of $\mathbb{D}|_{\mathcal{X}}$ such that $X_t \succ_i X_s$ for all $i \in N$. Because $\mathcal{D}|_{\mathcal{X}} = \mathcal{L}(\mathcal{X})$, there exists $R \in \mathcal{D}$ such that $X_t R|_{\mathcal{X}} X_s$, i.e., $R|_{\mathcal{X}} = \succsim_i$ for all $i \in N$. Let $R_N = (R, \ldots, R)$. Then, $R_N \in \mathbb{D}$, and by unanimity of f, $f(R_N) = r_1(R) \in X_t$. Therefore, the value of h at $R_N|_{\mathcal{X}}$ should be X_t, that is, $h(R_N|_{\mathcal{X}}) = X_t$. Because $R_N|_{\mathcal{X}} = \succsim_N$, $h(\succsim_N) = X_t$.

STEP 2: Let t be any element of $\{1, 2\}$. I prove that when $\mathbb{D}_{X_t} \neq \emptyset$, then g^t is unanimous. Let x be any element of X_t and let $\succsim_N$ be any element of $\mathbb{D}_{X_t}$ such that $r_1(\succsim_i) = x$ for all $i \in N$. Because $\succsim_N \in \mathbb{D}_{X_t}$, there exists $R_N \in \mathbb{D}$ such that $f(R_N) \in X_t$ and $R_N|_{X_t} = \succsim_N$.

I prove $f(R_N) = x$. Suppose to the contrary that $f(R_N) \neq x$. Unanimity of f implies that there exists $i' \in N$ such that $r_1(R_{i'}) \neq x$. Because x is the best alternative in X_t with respect to $R_{i'}$, it follows that $r_1(R_{i'}) \in X_s$, where s is the element of $\{1, 2\} \setminus \{t\}$. Then, there exists $i^* \in N$ such that $r_1(R_{i^*}) \in X_t$. (If not, then $X_s R_i|_{\mathcal{X}} X_t$ for all $i \in N$. By Step 1, h is unanimous, and hence $h(R_N|_{\mathcal{X}}) = X_s$. This implies $f(R_N) \in X_s$, which is a contradiction to the assumption $f(R_N) \in X_t$.) Because x is the most preferred alternative in X_t with respect to R_{i^*} and $r_1(R_{i^*}) \in X_t$, we have $r_1(R_{i^*}) = x$. Let R^* denote R_{i^*}, and let $R^*_N = (R^*, \ldots, R^*)$. By unanimity of f, $f(R^*_N) = x$ holds. Remember the assumptions $f(R_N) \in X_t$, $f(R_N) \neq x$, and x is the best alternative in X_t for every agent with respect to R_N. Consider the successive change of preferences from R_N to R^*_N. Recall that R^*_N consists of the common preferences R^* and at R^*, X_t is preferred to X_s. Because h is strategy-proof, at any stage of the successive change from R_N to R^*_N, the value of f is always in X_t. Because $f(R^*_N) = x$, at some stage, a social outcome changes from some $X_t \setminus \{x\}$ to x, which is a contradiction to strategy-proofness of f. Therefore, $f(R_N) = x$.

Because $x \in X_t$, it follows that $g^t(R_N|_{X_t}) = x$. Because $R_N|_{X_t} = \succsim_N$, $g^t(\succsim_N) = x$, and hence g^t is unanimous.

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