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The Optimality for Indirect Taxes

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Abstract

Since Frank P. Ramsey (1927) tackled an optimization problem for an ad valorem tax rate in a good market, he and his followers have made some tax rules from an "infinitesimal" tax rate believing such rules should be perfectly available, e. g., for "a tax of 500% on whisky." Ramsey also conjectured "the more complicated results ⋯ may well be valid under still wider conditions." However, we show here the limit of the 0/0 form renowned as the l'Hôpital's rule reveals his 'reciprocal elasticity rule' is invalid even for the "infinitesimal" rate. We also explore some closed-form solutions by taking a tax revenue elasticity of equilibrium quantity as a percentage-reduction rule for indirect taxes; our simpler results "valid under still wider conditions" imply that there exists no trade-off between efficiency and equity for good even from among more than two-good markets. (*JEL* H21)

These days, indirect taxes such as sales tax, value added tax, or commodity tax are important resources for governmental tax revenues in the world: *i. e.*, taxes on goods and services as percentage of total taxation are 17.4, 19.4, 29.0, and 30.3 in U.S., Japan, Germany, and U.K., respectively; and averagely 31.9 in all OECD countries in 2005.[1] However, it is a classic issue for economists and policy makers to design indirect taxes to maximize social welfare. Frank P. Ramsey (1927) established a path-breaking rule for

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[1]See, *e. g.*, OECD (2007) @pp.33-8 for details.

an ad valorem tax rate known as the 'Ramsey rule' or 'reciprocal elasticity rule.' Since then, as not only his setup but also his idea of minimizing a deadweight loss under a certain tax revenue was considered to be excellent, his tax rule has been adopted in a lot of textbooks for public finance, widely influencing students, scholars, and governments. So, we would like to make the following literature review on the rule before we start reexamining it.

The Literature

Suggested by Arthur C. Pigou, Frank P. Ramsey tackled to solve an optimization problem for ad valorem tax rates to answer under a certain tax revenue, what tax rates should be imposed on different goods maximizing social welfare. To do so, he built up a mathematical model with a few of parameters R, n, and so on where a government collects her tax revenue R from among n-good markets by maximizing social welfare or equivalently by minimizing deadweight losses in these markets. Then, he managed to make the 'reciprocal elasticity rule' from an "infinitesimal" tax rate or revenue.

According to Jerrold Marsden and Alan Weinstein (1985) @p.73,[2] the *Infinitesimal Calculus* was born in late 1960s, but Frank P. Ramsey (1927) @p.60 had concluded without any parametric closed-form solution "that the results about "infinitesimal" taxes can only claim to be approximately true for small taxes, how small depending on data which are not obtainable. It is perfectly possible that a tax of 500% on whisky could for the present purpose be regarded as small."

The '*Ramsey rule,*' or his equations (3) and (11) where we underline to make them different from ours,[3] insists that '*every optimal tax rate should be proportional to a sum of reciprocals of both price elasticities of demand and supply*' in each good market when she needs her tax revenue by R. So, his '*reciprocal elasticity rule*' makes another rule implying that high tax rates should be imposed on goods considered to be necessities like food with low

[2] See Jerome H. Keisler (2002) @p.33-76, and the last paragraph in our appendix C.

[3] As proved in appendix C, his equation (11) should be evaluated as our equation (C11).

price elasticities or inelastic demands. Because it is said that those goods consumers often have low income elasticities on them, if a high tax is levied on them, then the poor will be suffered from a larger burden than the rich. It clearly seems to be unfair to damage an interpersonal equity. When she did choose the "*Ramsey rule*s" to keep an economy efficient, it would lead to a loss of equity. Hence, it is not easy for her to operate such an indirect tax. This is a famous issue of a trade-off between efficiency and equity, which can be seen in the literature as well as in many standard textbooks in the world such as Richard A. Musgrave and Peggy B. Musgrave (1984) @p.312, Toshihiro Ihori (1996) @p.60, Louis Kaplow (2008) @p.146, and *etc.*

Arthur C. Pigou (1928, 1929, 1947) may be the first one who incorporated the "*Ramsey rules*" into a textbook as one chapter, in which he gave detailed, comprehensive, and intuitive explanations on "*his reciprocal elasticity rule and its implication as another rule.*" Thus, owing to Pigou's efforts, probably not by Frank P. Ramsey's original paper but the Pigou's textbook, a number of readers could catch up with what his original one meant as Ramsey (1927) did complex mathematics in his paper, which seems difficult to understand.

Paul A. Samuelson (1951) examined the Ramsey taxation problem. Since then, he has reprinted it in 1982 and again in 1986 with a brief introduction of his results: One of results must be close to Frank P. Ramsey's so called as the "Ramsey-Samuelson modified percentage-cut rule" (1951) @p.95; it is interesting to see in his paper that he does not have a closed-form solution, either, saying that "we can eliminate the Lagrangian multiplier $\cdots$, and write down the n relations determining the n unknowns, $(t_1, \cdots, t_n)$, as (8.7). $\cdots$ Ramsey's problem is formally solved by (8.7). However, the result is not very intuitive," *ibid.*, @p.87 where one had better determine the $(n+1)$-th unknown of the multiplier in the first place; otherwise, she or he has left the first-order condition unfinished as an optimality for indirect taxes.

In line with Frank P. Ramsey, many researchers seldom have any parametric closed-form solution, but examine his optimal taxation problem: *i.e.*, Robert L. Bishop (1968), Avinash K. Dixit (1970), Diamond and Mirrlees (1971), Joseph E. Stiglitz and Partha S. Dasgupta (1971), Martin S. Feldstein (1972), Anthony B. Atkinson and Joseph E. Stiglitz (1972, 1976, 1980), Robert Cooter (1978), Alan J. Auerbach (1985), Joseph E. Stiglitz (1986, 1987), and *etc.*[4] The "Ramsey rules" are also employed for an international commodity taxation in Michael Keen and David Wildasin (2004).

Purposes and Structure of this Paper

We cannot help having two main purposes here: One is to reexamine the "Ramsey rules;" the other is to find out some parametric closed-form solutions that no one has yet shown before in the literature because Frank P. Ramsey and his followers have made their rules from an "infinitesimal" tax rate since 1927 @p.60, believing without any closed-form solution that these rules could be perfectly valid for "a tax of 500% on whisky," and that "the more complicated results $\cdots$ may well be valid under still wider conditions."

To engage in them, the rest of our paper is organized as follows. First of all, section I presents the limit of the 0/0 form renowned as the l'Hôpital's rule discloses the "reciprocal elasticity rules" are illusions everywhere in the domains: That is, none of infinitesimal tax rates can be proportional to a sum of reciprocals of price elasticities of demand and supply at all even with a horizontal supply function because the sum is always too large to have the same order as infinitesimal candidates in the 0/0 form. Second, section II gives us a rule for an indirect taxation: *i.e.*, we take a tax revenue elasticity of equilibrium quantity as a percentage-reduction rule for indirect taxes, and explore several parametric examples in order to seek some closed-form solutions "valid under still wider conditions." Finally, section III concludes this paper. Our findings are totally different from the "Ramsey rules."

We put all proofs in appendix A; meanwhile, appendixes B and C are also prepared for the second-order condition and for a review of Ramsey (1927).

[4]Other papers are somehow related to the "Ramsey rules," or "his reciprocal elasticity rule and its implication as another rule." See, for example, Robin Boadway (1968), Avinash K. Dixit (1970), James A. Mirrlees (1972), and Perter A. Diamond (1975) in details.

I. The Ramsey Tax Problem

A. *The Model based upon Ramsey's Part III*

Just as Frank P. Ramsey (1927) @p.55-8, consider an optimization problem where a government imposes an ad valorem tax rate μ_r on n commodities with a running index r as $r = 1, 2, \cdots$, and n in order to collect her tax revenue by an amount of R. Let p_r and x_r be a price and a quantity of the r-th commodity, respectively, then assume in its commodity market that she faced not only an inverse demand function of $p_r = \phi_r(x_r)$ with a negative slope of $\phi_r'(x_r) \equiv \dfrac{d\,\phi_r(x_r)}{d\,x_r} < 0$ but also a supply function of $p_r = f_r(x_r)$ with a non-negative slope of $f_r'(x_r) \equiv \dfrac{d\,f_r(x_r)}{d\,x_r} \geq 0$, and so that she could define a vertical length λ_r between two functions independently on x_r as

$$(1) \quad \lambda_r = \lambda_r(x_r) \equiv \phi_r(x_r) - f_r(x_r) \geq 0$$

in a domain of

$$(2) \quad 0 < x_r \leq \bar{x}_r$$

where $\bar{x}_r$ is an initial equilibrium quantity such that $\phi_r(\bar{x}_r) = f_r(\bar{x}_r)$ when there is no taxation on the r-th commodity with the tax rate of $\mu_r = 0$.

It is known that if some tax rate $\mu_r > 0$ is imposed, then geometrically speaking, the supply function $p_r = f_r(x_r)$ shifts up to $p_r = (1 + \mu_r)f_r(x_r)$ along the inverse demand function $p_r = \phi_r(x_r)$, so that a new equilibrium quantity x_r after tax should satisfy an equation of $\phi_r(x_r) = (1 + \mu_r)f(x_r)$. Thus, equation (1) or a difference between two functions λ_r becomes

$$(3) \quad \lambda_r = \phi_r(x_r) - f_r(x_r) = \mu_r f_r(x_r),$$

then the ad valorem tax rate μ_r can be expressed as

$$(4) \quad \mu_r = \frac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \frac{\lambda_r(x_r)}{f_r(x_r)} = \frac{\lambda_r}{f_r} \geq 0$$

in terms of the quantity x_r. So, it is easy to see by taking the derivative of equation (4) with respect to x_r in the domain of equation (2) as

$$(5) \quad \mu_r' \equiv \frac{d\,\mu_r}{d\,x_r} = \frac{\phi_r'(x_r)f_r(x_r) - \phi_r(x_r)f_r'(x_r)}{\{f_r(x_r)\}^2} < 0$$

that the tax rate μ_r is monotonously decreasing with respect to x_r owing to the signs of slopes $\phi_r' < 0$ and $f_r' \geq 0$, and so that the government can treat the quantity x_r as her choice variable instead of the tax rate μ_r.[5]

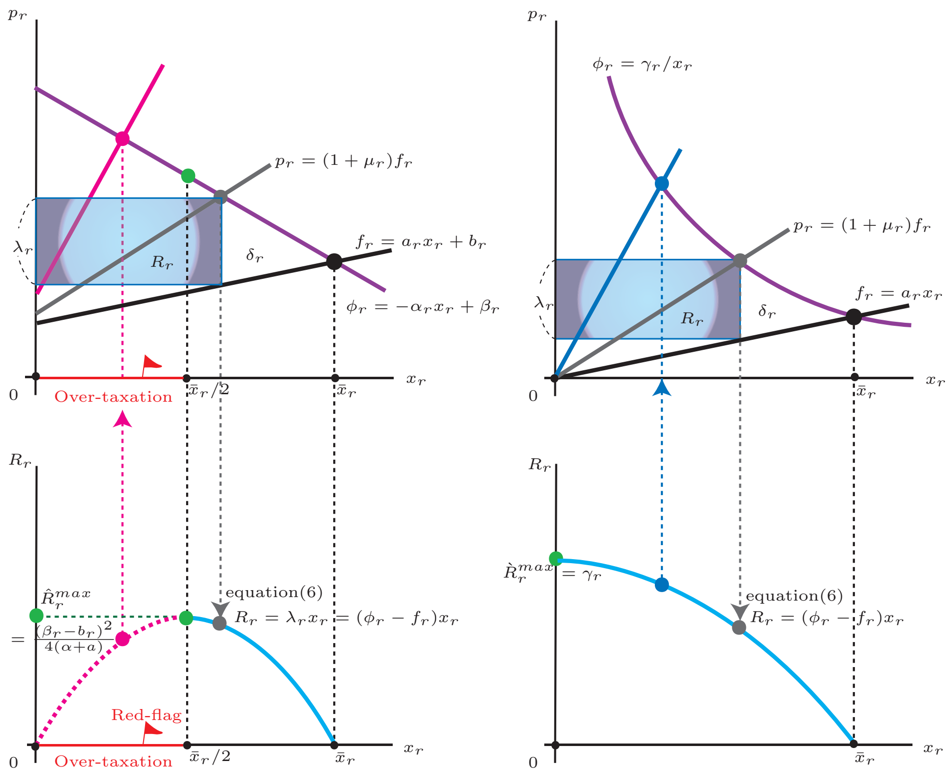

Figure 1: A Rectangular Area with an Ad Valorem Tax Rate μ_r

It is also known as shown in figure 1 that if the tax rate $\mu_r > 0$ is levied, then a rectangular area, namely, a product of the length λ_r in equation (1) or (3) and a width of the quantity x_r makes a tax revenue of

$$(6) \qquad R_r \equiv \lambda_r \, x_r = \{\phi_r(x_r) - f_r(x_r)\} \, x_r \geq 0.$$

To find out a potentially maximal amount R_r^{max} of the tax revenue without her tax constraint R, we usually attempt the following two ways to do so: The first way is to take the derivative of equation (6) with respect to x_r and

[5]In the case of a unit tax υ_r, we can replace equations (1) or (3), (4), and (5) by $\lambda_r = \phi_r(x_r) - f_r(x_r) = \upsilon_r \geq 0$, $\upsilon_r = \phi_r(x_r) - f_r(x_r) \geq 0$, and $\upsilon_r' \equiv \frac{d\,\upsilon_r}{d\,x_r} = \phi_r'(x_r) - f_r'(x_r) < 0$, respectively since the supply function $p_r = f_r(x_r)$ shifts up parallel to $p_r = f_r(x_r) + \upsilon_r$ by the unit tax υ_r along the inverse demand function $p_r = \phi_r(x_r)$.

set it equal to zero if we can as

$$(7)\qquad \frac{d\,R_r}{d\,x_r} \equiv \lambda'_r\, x_r + \lambda_r = 0$$

where λ'_r is the derivative of equation (1) or (3) with respect to x_r of

$$(8)\qquad \lambda'_r \equiv \frac{d\,\lambda_r(x_r)}{d\,x_r} = \phi'_r(x_r) - f'_r(x_r) < 0 \qquad \text{for } r = 1, 2, \cdots, n$$

anywhere in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$ due to the signs of slopes $\phi'_r < 0$ and $f'_r \geq 0$. Because of equation (7), we acquire the maximum amount $R_r^{max} = \hat{R}_r^{max}$ at $x_r = x_r^{max}$ such that $\dfrac{-\lambda_r}{\lambda'_r\, x_r} = 1$ in its domain ($0 < x_r \leq \bar{x}_r$). For example, the government may levy the potentially maximal tax of $\hat{R}_r^{max} \equiv \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$ as her revenue at $x_r^{max} \equiv \dfrac{\beta_r - b_r}{2\,(\alpha_r + a_r)}$ on an affine market where she faces not merely an inverse demand function $\phi_r \equiv -\alpha_r\, x_r + \beta_r$ but also a supply function $f_r \equiv a_r\, x_r + b_r$ with parameters $\alpha_r > 0$, $\beta_r > b_r \geq 0$, and $a_r \geq 0$ but $a_r = b_r \neq 0$ simultaneously; on the other hand, the second one is to take the limit of equation (6) as x_r goes to the origin 0 from the right hand side denoted by $^+$ when she meets in a market that

$$(9)\qquad \frac{d\,R_r}{d\,x_r} \equiv \lambda'_r\, x_r + \lambda_r < 0$$

everywhere in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$. From equation (6), in this case, she attains its potential maximum tax revenue $R_r^{max} = \grave{R}_r^{max}$ as

$$(10)\qquad \lim_{x_r \to 0^+} R_r \to R_r^{max} = \grave{R}_r^{max} > 0$$

as the length λ_r in equation (1) or (3) reaches its maximum at $x_r = 0$ from equation (8). A demand function $\phi_r \equiv \dfrac{\gamma_r}{x_r}$, $\gamma_r > 0$, *e. g.*, yields the maximum as $\grave{R}_r^{max} \equiv \gamma_r$ to her from its limit of $\lim_{x_r \to 0^+} R_r \to \dfrac{\gamma_r}{x_r}\, x_r - f_r(0)\, 0 = \gamma_r$.

Because equation (6), sometimes a mountain shape as the Arther Laffer curve, can give us at most one summit of the maximal revenue $\hat{R}_r^{max}$ in the domain of equation (2): $0 < x_r \leq \bar{x}_r$, in addition to equations (7) and (9) of

$$(11)\qquad \frac{d\,R_r}{d\,x_r} \equiv \lambda'_r\, x_r + \lambda_r \leq 0,$$

the mountain hidden in equation (6) should possess a quasi-concavity as

$$(12)\quad \frac{d^2 R_r}{d\,x_r^2} \equiv \lambda_r''\, x_r + 2\,\lambda_r' \leq 0.$$

So, equation (12) has an important role, *e.g.*, for the second-order condition as seen in appendix B. It is worth noticing here that for a tax problem with an amount $R_r^{\exists}$ less than the potential maximum R_r^{max} in equation (6): That is, for $R_r^{max} > R_r^{\exists} \geq 0$, equation (9) prevails to equation (7) or

$$(13)\quad \frac{d\,R_r}{d\,x_r} \equiv \lambda_r'\, x_r + \lambda_r < 0$$

at $x_r = x_r^{R_r^{\exists}}$ in the domain of equation (2) as it always happens somewhere on the right of the maximum in equation (6) as $0 \leq x_r^{max} < x_r^{R_r^{\exists}} = x_r \leq \bar{x}_r$.

Moreover, it is well known as seen in figure 1 that if the rate $\mu_r > 0$ is levied, an area δ_r so called a deadweight loss appears as

$$(14)\quad \delta_r \equiv \int_{x_r}^{\bar{x}_r} \lambda_r\, ds_r = \int_{x_r}^{\bar{x}_r} \{\phi_r(s_r) - f_r(s_r)\}\, ds_r.$$

Therefore, the government can take the quantity x_r as a choice variable and maximize an objective function $\mathcal{U}$ of a sum of u_r $(\equiv -\delta_r)$ subject to a constraint given by the constant tax revenue R: That is, she

$$\underset{x_1,\, x_2,\, \cdots,\, x_n}{\text{maximizes}}\ \mathcal{U} \equiv \sum_{r=1}^{n} u_r \text{ subject to } R = \sum_{r=1}^{n} R_r;\ \text{or she}$$

$$(15)\quad \underset{x_1,\, x_2,\, \cdots,\, x_n,\, \kappa}{\text{maximizes}}\ \mathcal{L} \equiv -\sum_{r=1}^{n} \int_{x_r}^{\bar{x}_r} \lambda_r\, ds_r + \kappa\,(R - \sum_{r=1}^{n} \lambda_r\, x_r)$$

where $\mathcal{L}$ is a Lagrange function with a multiplier κ. Taking partial derivatives of equation (15) with respect to choice variables, $x_1, x_2, \cdots, x_n$, and κ yields to us a set of the following $(n+1)$ equations for the first-order condition: [6]

$$\frac{\partial\,\mathcal{L}}{\partial\,x_r} = \lambda_r - \kappa(\lambda_r + \lambda_r'\, x_r) = 0 \qquad \text{for } r = 1, 2, \cdots, n;$$

$$\frac{\partial\,\mathcal{L}}{\partial\,\kappa} = R - \sum_{r=1}^{n} \lambda_r\, x_r = 0.$$

[6]See appendix B for the second-order condition in our program of equation (15): That is, a stationary value of the Lagrange function $\mathcal{L}$ from parametric or numerical solutions, say $\bar{\mathcal{L}}$ needs to be tested against a second-order condition called a bordered Hessian. For this Hessian, *e.g.*, see Alpha C. Chiang (1984) @pp.379-87, who gives us a word of caution on the multiplier κ.

To summarise the above equations, let us replace the multiplier κ by $-K$ for a while, as seen in Frank P. Ramsey (1927) @p.50, then we have

$$(16) \quad K = K_r = K_r(x_r) \equiv -\frac{\lambda_r}{\lambda_r + \lambda_r'\, x_r} \qquad \text{for } r = 1, 2, \cdots, n,$$

$$(17) \quad R = \sum_{r=1}^{n} R_r = \sum_{r=1}^{n} \lambda_r\, x_r,$$

in which $K_1 = K_2 = \cdots = K_n$ is the condition of the equal-percentage-reduction rule called by Paul A. Samuelson (1951) @p.77.[7]

B. *Proportionality in* $0 < x_r < \bar{x}_r$

It is obvious in an open set of $0 < x_r < \bar{x}_r$ stemmed from equation (2) that a percentage-reduction $K_r(x_r)$ in equation (16) cannot be proportional to the others for all $r = 1,\ 2,\ \cdots,\ n$ since the limit of the reduction $K_r(x_r)$ never converges to zero as x_r approaches zero from the right hand (0^+) :

$$(18) \quad \lim_{x_r \to 0^+} K_r(x_r) = -\frac{\lambda_r}{\lambda_r + \lambda_r' \times 0} = -1 \neq 0.$$

Mathematically speaking, equation (18) means that nobody can imagine any linear relationship among them (percentage-reductions $K_1,\ K_2,\ \cdots,\ K_n$ and their quantities $x_1,\ x_2,\ \cdots,\ x_n$). So, nobody should have put them into straight lines going through the origin 0 even as an approximation ($\approx$) like

$$(19) \quad K = K_r = K_r(x_r) = -\frac{\lambda_r}{\lambda_r + \lambda_r'\, x_r} \not\approx c_r\, x_r$$

with a constant coefficient of c_r everywhere in the set of $0 < x_r < \bar{x}_r$ for all $r = 1,\ 2,\ \cdots,\ n$. In fact, moreover, an identical value of the multiplier K, say $K = K^*$ to every percentage-reduction K_r, does not always provide us with a linearity among them because the function of $K^* = K_r(x_r)$ does not always go through the origin 0 at $x_r = 0$. Furthermore, as an application

[7] For the total-differential approach, equation (16) becomes $\frac{\partial\, u_r/\partial\, x_r}{\partial\, R_r/\partial\, x_r} = \frac{d\,\delta_r/d\,x_r}{d\,R_r/d\,x_r} = \frac{d\,\delta_r}{d\,R_r} = \frac{\lambda_r}{\lambda_r+\lambda_r'\,x_r} = -K$ that corresponds to equation (3) (where we underline to make it different from ours) in Frank P. Ramsey (1927) for all $r = 1,\ 2,\ \cdots,\ n$. Recall that the multiplier $\kappa = -K$ is hardly treated as a parameter nor a constant value but as a choice variable to be solved whose solution reflects $\frac{d\,\mathcal{L}}{d\,R}$. See Alpha C. Chiang (1984) @p.375 again.

of those rules used in equations (18) and (19) in order to vanish an illusion of proportionality or linearity, it is easy to see almost everywhere in the set of $0 < x_r < \bar{x}_r$ except in the neighborhood of $\bar{x}_r$ that each indirect tax is not proportional to any function of x_r, say $\sigma_r = \sigma_r(x_r)$, *i. e.*, as $x_r \to 0^+$, not merely the limit of an ad valorem tax rate μ_r but also that of a unit tax υ_r never converges to zero but a positive number due to its monotone decreasing with respect to x_r as seen in equations (4), (5), and footnote 5:

$$(20) \quad \lim_{x_r \to 0^+} \mu_r = \lim_{x_r \to 0^+} \mu_r(x_r) = \frac{\phi_r(0) - f_r(0)}{f_r(0)} > 0;$$

$$(21) \quad \lim_{x_r \to 0^+} \upsilon_r = \lim_{x_r \to 0^+} \upsilon_r(x_r) = \phi_r(0) - f_r(0) > 0,$$

which means an indirect tax never passes through the origin 0 at $x_r = 0$.

C. *Proportionality in the Neighborhood of* $x_r = \bar{x}_r$

In the meantime, it is also obvious at every point close enough to $x_r = \bar{x}_r$ in the set of $0 < x_r < \bar{x}_r$ for all $r = 1, 2, \cdots, n$ that infinitesimal percentage-reductions $K_r(x_r)$ in equation (16) can be proportional each other because its value of $K_r(x_r)$ must be small as $K_r(x_r) \approx 0$ in taking the limit as x_r tends to the initial one of $\bar{x}_r$ from the left hand side denoted by $^-$. As it can be seen in a market at $x_r = \bar{x}_r$ before tax that we have $\phi_r(\bar{x}_r) = f_r(\bar{x}_r)$ from the demand = supply condition or $\phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0$, the following left-hand limit gives us that

$$(22) \quad \lim_{x_r \to \bar{x}_r^-} K_r(x_r) = -\frac{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)}{\phi_r(\bar{x}_r) - f_r(\bar{x}_r) + \lambda'_r(\bar{x}_r) \times \bar{x}_r} = 0$$

with a non-zero product of equation (8) or λ'_r (< 0) and $\bar{x}_r$ (> 0). It is easy for us to seek a candidate of another infinitesimal function of x_r around $\bar{x}_r$: For instance, equations (1) or (3), (4), (6), and footnote 5 drive us

$$(23) \quad \lim_{x_r \to \bar{x}_r^-} \lambda_r = \lim_{x_r \to \bar{x}_r^-} \lambda_r(x_r) = \phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0;$$

$$(24) \quad \lim_{x_r \to \bar{x}_r^-} \mu_r = \lim_{x_r \to \bar{x}_r^-} \mu_r(x_r) = \frac{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)}{f_r(\bar{x}_r)} = 0;$$

$$(25) \quad \lim_{x_r \to \bar{x}_r^-} R_r = \lim_{x_r \to \bar{x}_r^-} R_r(x_r) = \{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}\, \bar{x}_r = 0;$$

$$(26) \quad \lim_{x_r \to \bar{x}_r^-} \upsilon_r = \lim_{x_r \to \bar{x}_r^-} \upsilon_r(x_r) = \phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0.$$

So, *e. g.*, a revenue R_r is proportional to ($\propto$) an ad valorem tax rate μ_r as $R_r \propto \mu_r$ with a straight line of $R_r = c_r\,\mu_r$ nearby at $x_r \leq \bar{x}_r$ for an infinitesimal $\mu_r \geq 0$ where a constant coefficient $c_r \equiv \phi_r(\bar{x}_r)\,\bar{x}_r$ is given as follows:[8]

$$(27)\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{R_r}{\mu_r} = \frac{\{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}\,\bar{x}_r}{\{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}/f_r(\bar{x}_r)} = f_r(\bar{x}_r)\,\bar{x}_r \equiv c_r,$$

which also supplies us with the famous Guillaume de l'Hôpital's rule as

$$(28)\quad \lim_{x_r \to \bar{x}_r^-} \frac{R_r}{\mu_r} = \lim_{x_r \to \bar{x}_r^-} \frac{\{R_r(x_r) - R_r(\bar{x}_r)\}/(x_r - \bar{x}_r)}{\{\mu_r(x_r) - \mu_r(\bar{x}_r)\}/(x_r - \bar{x}_r)} = \frac{R_r'(\bar{x}_r)}{\mu_r'(\bar{x}_r)}.$$

D. *Illusions upon the Ramsey Tax Rules*

Consequently, all we need is to show if a function of x_r, say $\sigma_r = \sigma_r(x_r)$ is convergent to zero as x_r approaches $\bar{x}_r$ in order that an infinitesimal σ_r gets proportional to sufficiently small indirect taxes, μ_r and υ_r for all $r = 1, 2, \cdots, n$. Otherwise, any $\sigma_r(x_r)$ is dominated by the limit like equations (18), (20), and (21) since every real number x_r in the set of $0 < x_r < \bar{x}_r$ is now way far from the origin 0. Due to equations (20) and (21), needless to say, none of $\sigma_r(x_r)$ should be proportional to an ad valorem tax rate $\mu_r(x_r)$ nor a unit tax $\upsilon_r(x_r)$ almost everywhere except in the neighborhood of $x_r = \bar{x}_r$.

In this sense, denote a sum of reciprocals of a price elasticity of demand and that of supply by $\sigma_r = \sigma_r(x_r)$ for $r = 1, 2, \cdots, n$, namely,

$$(29)\quad \sigma_r \equiv \frac{1}{\rho_r} + \frac{1}{\varepsilon_r} = -\frac{\phi_r'(x_r)\,x_r}{\phi_r(x_r)} + \frac{f_r'(x_r)\,x_r}{f_r(x_r)}$$

where ρ_r is a price elasticity of demand or $\rho_r \equiv -\dfrac{\phi_r(x_r)}{\phi_r'(x_r)\,x_r}$ and ε_r is that of supply or $\varepsilon_r \equiv \dfrac{f_r(x_r)}{f_r'(x_r)\,x_r}$, then we have the following proposition:[9]

PROPOSITION 1: *The Ramsey tax rule, which claims anywhere even close enough to* $x_r = \bar{x}_r$ *that an ad valorem tax rate* μ_r *could be proportional to* ($\propto$) *the sum of reciprocals of price elasticities of demand and supply* $\sigma_r(x_r)$

[8] For a sufficiently small unit tax $\upsilon_r \geq 0$, a revenue R_r is proportional to ($\propto$) its υ_r as $R_r \propto \upsilon_r$ at $x_r \leq \bar{x}_r$ or $R_r = c_r\,\upsilon_r$ with a constant coefficient c_r from the left-hand limit of $\frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{R_r}{\upsilon_r} = \bar{x}_r \equiv c_r$. See Alpha C. Chiang (1984) @pp.428-30 for the $\frac{0}{0}$ form.

[9] See appendix C for a review about a few equations in Frank P. Ramsey (1927).

$= \dfrac{1}{\rho_r} + \dfrac{1}{\varepsilon_r}$, *is an illusion: i. e.,* $\mu_r \not\propto \sigma_r(x_r)$ *even for a sufficiently small* μ_r*; by contraries,* $\mu_r \propto K_r$ *in the neighborhood of* $x_r \leq \bar{x}_r$ *through a linear relationship* $\mu_r = \sigma_r(\bar{x}_r)\, K_r$ *for infinitesimal tax rates* $\mu_r \geq 0$ *from equation (4) and percentage-reductions* K_r *from equation (16) for all* $r = 1,\, 2,\, \cdots,\, n$*.*

Proof. See appendix A for this proof. □

We can see in appendix A that even with a horizontal supply function, a case of slope-zero or $f_r'(x_r) = 0 = \dfrac{1}{\varepsilon_r}$, our proposition 1 is robust, and from equation (C14) of appendix C that proportionality with exactly the same coefficients $\sigma_r(\bar{x}_r)$ as $\mu_r = \sigma_r(\bar{x}_r)\, \theta_r$ hold among these rates μ_r and length elasticities θ_r in equation (C3) nearby at $x_r \leq \bar{x}_r$ for all $r = 1,\, 2,\, \cdots,\, n$.

Anyway, equation (A4) gives us another $\dfrac{0}{0}$ form like equation (27) as well as the l'Hôpital's rule like equation (28) as

$$(30)\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{K_r} = \lim_{x_r \to \bar{x}_r^-} \frac{\{\mu_r(x_r) - \mu_r(\bar{x}_r)\}/(x_r - \bar{x}_r)}{\{K_r(x_r) - K_r(\bar{x}_r)\}/(x_r - \bar{x}_r)} = \frac{\mu_r'(\bar{x}_r)}{K_r'(\bar{x}_r)} = c_r$$

owing to $\mu_r(\bar{x}_r) = K_r(\bar{x}_r) = 0$ from equations (24) and (22). Equation (30) converges to the constant value c_r as shown in equations (A3) and (A4) as

$$(\text{A3})\quad c_r = \lim_{x_r \to \bar{x}_r^-} \sigma_r(x_r) = \sigma_r(\bar{x}_r) = -\frac{\lambda_r'(\bar{x}_r)\, \bar{x}_r}{f_r(\bar{x}_r)} = \frac{1}{\rho_r(\bar{x}_r)} + \frac{1}{\varepsilon_r(\bar{x}_r)},$$

so that by using the unit-less quantity q in equation (A5), one may estimate a rate $\tilde{\mu}_r$ approximately with an infinitesimal percentage-reduction K_r as

$$(31)\quad \tilde{\mu}_r = \tilde{\mu}_r(K_r) \equiv \sigma_r(\bar{x}_r)\, K_r(\bar{x}_r\, q)$$

at $q \equiv \dfrac{x_r}{\bar{x}_r} \leq 1$ when x_r is close enough to $\bar{x}_r$ or equal to it. Even for any infinitesimal μ_r, thus, we have $\mu_r \not\propto \sigma_r$, but $\mu_r \propto K_r$ instead: *i. e.*, any ad valorem tax rate μ_r is never proportional to σ_r, the sum of reciprocals of price elasticities of demand and supply; but to the percentage-reduction K_r in the neighborhood of $\bar{x}_r$ to retain the nil term of $[\{\lambda_r(x_r)\}] \approx 0$ in equation (A4). Neither is a unit tax υ_r.[10]

[10] The $\frac{0}{0}$ form of equations (26) to (22) or $\lim_{x_r \to \bar{x}_r^-} \frac{\upsilon_r}{K_r} = f_r(\bar{x}_r)\, \frac{\lambda_r(\bar{x}_r)}{f_r(\bar{x}_r)}\, \frac{[\{\lambda_r(x_r)\}] + \lambda_r'(\bar{x}_r)\, \bar{x}_r}{-\lambda_r(\bar{x}_r)} =$

COROLLARY 1: *The Ramsey tax ratio rule, which claims even nearby at* $x_r = \bar{x}_r$ *that a ratio of ad valorem tax rates* $\dfrac{\mu_i}{\mu_j}$ *could be proportional to* ($\propto$) *that of the sum of reciprocals of price elasticities* $\dfrac{\sigma_i(x_i)}{\sigma_j(x_j)}$, *is a hallucination: i. e.,* $\dfrac{\mu_i}{\mu_j} \not\propto \dfrac{\sigma_i(x_i)}{\sigma_j(x_j)}$ *at all; on the contrary,* $\dfrac{\mu_i}{\mu_j} \propto \dfrac{K_i(x_i)}{K_j(x_j)}$ *such as* $\dfrac{\mu_i(x_i)}{\mu_j(x_j)} = \dfrac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)}\,\dfrac{K_i(x_i)}{K_j(x_j)}$ *for* $x_r \leq \bar{x}_r$ *but very close to* $\bar{x}_r$ *corresponding to infinitesimal tax rates of* $\mu_r \geq 0$ *from equation (4) over percentage-reductions* K_r *from equation (16) for all* i, j, *and* $r \in \{r \mid r = 1, 2, \cdots, n\}$.

Proof. It is obvious from proposition 1, but see appendix A for this. □

It is also easy for us to show for all i, j, and $r \in \{r \mid r = 1, 2, \cdots, n\}$ from footnote 10 that we can appreciate the following limit of a ratio of

$$(32) \quad \lim_{q \to 1^-} \frac{\upsilon_i / K_i}{\upsilon_j / K_j} = \lim_{q \to 1^-} \frac{f_i(\bar{x}_i\, q)}{f_j(\bar{x}_j\, q)}\,\frac{\sigma_i(\bar{x}_i\, q)}{\sigma_j(\bar{x}_j\, q)} = \frac{f_i(\bar{x}_i)}{f_j(\bar{x}_j)}\,\frac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)} \equiv c_{ij}^{\upsilon},$$

as the unit-less q tends to unity from the left in equation (A5), $0 < q = q_r \equiv \dfrac{x_r}{\bar{x}_r} \leq 1$. So, we can see nearly at $q = 1$ with negligible length of equation (1) or (3) of $[\{\lambda_r(\bar{x}_r\, q)\}] \approx 0$ that a ratio of unit taxes $\dfrac{\upsilon_i}{\upsilon_j} \not\propto \dfrac{\sigma_i(\bar{x}_i\, q)}{\sigma_j(\bar{x}_j\, q)}$, a ratio of the sums from equation (29); but the ratio of them $\dfrac{\upsilon_i}{\upsilon_j} \propto \dfrac{K_i(\bar{x}_i\, q)}{K_j(\bar{x}_j\, q)}$, a ratio of percentage-reductions with a linear relation $\dfrac{\upsilon_i(\bar{x}_i\, q)}{\upsilon_j(\bar{x}_j\, q)} = c_{ij}^{\upsilon}\,\dfrac{K_i(\bar{x}_i\, q)}{K_j(\bar{x}_j\, q)}$ where c_{ij}^{υ} is given by equation (32) as a constant coefficient for infinitesimal unit tax $\upsilon_r \geq 0$ upon percentage-reductions K_r very nearby at $q \leq q_r$.

One may still have a daydream in corollary 1 that equations (A6) and (32) could have implicated in a ratio of optimal indirect taxes as $\dfrac{\mu_i^*}{\mu_j^*}$ or $\dfrac{\upsilon_i^*}{\upsilon_j^*}$ because their numerators' K_i and denominators' K_j are seemingly canceled out by a common infinitesimal percentage-reduction $K^* = K_r(x_r^*) \neq 0$ given somewhere at $x_r = x_r^* < \bar{x}_r$ close enough to $\bar{x}_r$ or at $q = q_r^* < 1$, but they have implicated in that of rates of change of them as $\dfrac{d\,\mu_i}{d\,\mu_j}$ or $\dfrac{d\,\upsilon_i}{d\,\upsilon_j}$ because

$f_r(\bar{x}_r)\,\sigma_r(\bar{x}_r) \equiv c_r$ where $[\{\lambda_r(x_r)\}] \approx 0$ again. A unit tax υ_r is also proportional to ($\propto$) a percentage-reduction K_r as $\upsilon_r \propto K_r$ through $\upsilon_r = c_r\, K_r$ with the above constant $c_r \equiv f_r(\bar{x}_r)\,\sigma_r(\bar{x}_r)$ in the neighborhood of $x_r \leq \bar{x}_r$ for an infinitesimal unit tax of $\upsilon_r \geq 0$ for all $r = 1, 2, \cdots, n$. Then, in this case, $\upsilon_r \not\propto \sigma_r(\bar{x}_r) = \frac{1}{\rho_r(\bar{x}_r)} + \frac{1}{\varepsilon_r(\bar{x}_r)}$, either.

those infinitesimal variables are usually, mathematically speaking, denoted by $d\,K_r$, $d\,\mu_r$, $d\,\upsilon_r$, and so on for all i, j, and $r \in \{r \mid r = 1,\, 2,\, \cdots,\, n\}$. To see it, *e.g.*, recall equation (30) for the l'Hôpital's rule, which is tractable for us to convey it into

$$(33) \quad \frac{\mu_r'(\bar{x}_r)}{K_r'(\bar{x}_r)} = \left.\frac{d\,\mu_r/d\,x_r}{d\,K_r/d\,x_r}\right|_{x_r=\bar{x}_r} = \left.\frac{d\,\mu_r}{d\,K_r}\right|_{x_r=\bar{x}_r} = \left.\frac{d\,\mu_r}{d\,K_r}\right|_{\mu_r=K_r=0} = c_r.$$

The straight line $\mu_r = \sigma_r(\bar{x}_r)\,K_r$ in equations (30), (31), and (A4) is nothing but a tangent line at the origin (0, 0) of $K_r = \mu_r = 0$ in the K_r-μ_r plane, then its total derivative is calculated as $d\,\mu_r = \sigma_r(\bar{x}_r)\,d\,K_r$. Let $\mu_r^* = \mu_r(x_r^*)$ and $K^* = K_r(x_r^*)$ be an optimal ad valorem tax rate and its corresponding percentage-reduction, respectively. It is quite well known in calculus that the derivative is the limit of a difference quotient:[11] That is, a secant line, which

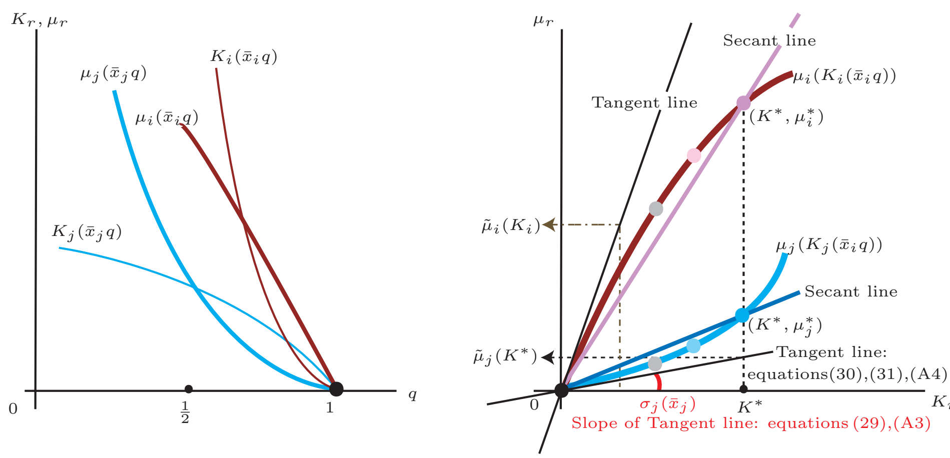

Figure 2: Relationships among Secant Lines and Tangent Lines

is also a straight line that passes through two points like (0, 0) and $(K^*,\, \mu_r^*)$ in figure 2, tends close to the tangent line as the secant point $(K^*,\, \mu_r^*)$ tends close to (0, 0), or in the Gottfried Wilhelm von Leibniz's notation that when a difference $\Delta\,K^* \equiv K^* - 0 = K^*$ became the infinitesimal $d\,K$ or $d\,K_r$, a difference $\Delta\,\mu_r^* \equiv \mu_r^* - 0 = \mu_r^*$ simultaneously became the infinitesimal $d\,\mu_r$ and their difference quotient of $\dfrac{\Delta\,\mu_r^*}{\Delta\,K^*} = \dfrac{\mu_r^* - 0}{K^* - 0} = \dfrac{\mu_r^*}{K^*}$ became $\dfrac{d\,\mu_r}{d\,K}$, which were not an approximation to the derivative any more but exactly the same

[11]See, *e.g.*, Jerrold Marsden and Alan Weinstein (1985) @pp.49-75 for the limit of a difference quotient, a secant line, the Leibniz's notation, and so forth.

as the derivative itself. In addition to this Leibniz's notation, remind any ratio $\dfrac{\mu_r^*}{K^*}$ of the secant line belongs to such an average function around (0, 0) that every ratio of ratios like $\dfrac{\mu_i^*}{\mu_j^*} = \dfrac{\mu_i^*}{K^*}\dfrac{K^*}{\mu_j^*}$ cannot even estimate a precise value from it, and so that one had better use our equation (4) to do so if she or he wants a precise one. Thus, any ratio of optimal indirect taxes such as $\dfrac{\mu_i^*}{\mu_j^*}$ and $\dfrac{\upsilon_i^*}{\upsilon_j^*}$ with an "infinitesimal" K^*, say $d\,K^*$ given without solving the multiplier $\kappa = (-K)$ no longer implicates in the optimal ratio of them but just an approximation to its derivative like $\dfrac{d\,\mu_i^*}{d\,\mu_j^*} = \dfrac{d\,\mu_i^*}{d\,K^*}\dfrac{d\,K^*}{d\,\mu_j^*}$ at most.

Finally, what we have shown here is the Ramsey tax rules unfortunately told us for eight decades that a variable is proportional to ($\propto$) a coefficient: *i.e.*, they should not have misled us into believing as if $y \propto 2$ in regard to a tangent line of $y = 2\,x$ with two variables x, y and a constant coefficient c_j = 2; which may represent the j-th good market in figure 2 with a demand function $\phi_j = \dfrac{\gamma_j}{x_j}$, $\gamma_j > 0$, a supply function $f_j = a_j\,x_j$, $a_j > 0$, these price elasticities $\rho_j = \varepsilon_j = 1$, and so $c_j = \sigma_j \equiv \dfrac{1}{\rho_j} + \dfrac{1}{\varepsilon_j} = 2$ from equation (29).

II. A Rule for Indirect Taxes

A. *A Treatment for the Multiplier* κ

So far, we have shown not only the limit of the $\dfrac{0}{0}$ form but also that of the $\dfrac{0/0}{0/0}$ form discloses in proposition 1 and corollary 1, respectively, that Frank P. Ramsey (1927) and his followers' tax rules are illusions: That is, anywhere in the domain none of indirect taxes can be proportional to a sum of reciprocals of a price elasticity of demand and that of supply even with a horizontal supply function, so that we can observe no trade-off between efficiency and equity even from among more than two good markets.

One of the reasons why they have made those illusionary rules is caused by an unusual treatment for a Lagrange-multiplier $\kappa = -K$. As mentioned in footnote 7 and section I.B, the multiplier κ is usually neither a parameter nor a constant value but a choice variable to be solved. Thus, if they solved a subset of the first-order condition or equation (16) itself with respect to the other n choice variables x_1, x_2, $\cdots$, and x_n in terms of only the multiplier κ in order to substitute them into the constraint of the constant tax revenue R or equation (17), then they could not have provided us with such illusions upon indirect taxes.

To see this procedure, let us rewrite equations (16) and (17) or summarized first-order condition here:

$$(16) \quad \kappa = -K = \frac{\lambda_r}{\lambda_r + \lambda_r' \, x_r} \qquad \text{for } r = 1, 2, \cdots, n;$$

$$(17) \quad R = \sum_{r=1}^{n} R_r = \sum_{r=1}^{n} \lambda_r \, x_r .$$

First of all, not by eliminating κ from equation (16) but by solving n equations with respect to x_r in terms of only κ for $r = 1, 2, \cdots, n$, we ought to have n functions x_r of κ as

$$(34) \quad x_r = x_r(\kappa) \qquad \text{for } r = 1, 2, \cdots, n.$$

Next, by plugging up each equation (34) in both λ_r and x_r of equation (17), we obtain a function of κ or

$$(35) \quad R = \sum_{r=1}^{n} \lambda_r(x_r(\kappa)) \, x_r(\kappa) = \sum_{r=1}^{n} R_r(\kappa),$$

which provides us with at least one root to be examined against a second-order condition of a bordered Hessian that we put in our appendix B.

Alternatively, so as to earn equation (34) which interfaces equations (16) and (17), we may calculate each market's tax revenue R_r of equation (17) in the first place because the multiplier κ in equation (16) can be interpreted as a percentage-reduction of a tax revenue R_r elasticity of equilibrium quantity x_r after tax that reflects R_r increases 1 % causing x_r to fall $|\kappa|$% as follows:

$$(36) \quad \kappa = \frac{\lambda_r \, x_r / x_r}{\lambda_r + \lambda_r' \, x_r} = \frac{R_r / x_r}{d\, R_r / d\, x_r} = \frac{d\, x_r / x_r}{d\, R_r / R_r} \qquad \text{for } r = 1, 2, \cdots, n;$$

which ends up with equation (35). Besides, it is easy to compute from equations (36) and (A5) that the multiplier κ is put into another tax revenue R_r

elasticity of equilibrium unit-less quantity $q = \dfrac{x_r}{\bar{x}_r}$ after tax or

$$(37) \quad \kappa = \frac{d\,x_r/x_r}{d\,R_r/R_r} = \frac{d\,q/q}{d\,R_r/R_r} = \frac{R_r/q}{d\,R_r/d\,q} \qquad \text{for } r = 1, 2, \cdots, n$$

since we have a sufficiently small $d\,x_r = \bar{x}_r\,d\,q$ as well as $x_r = \bar{x}_r\,q$ according to equation (A5). Needless to say in this case, we have n functions q of κ as $q = q(\kappa)$ for $r =$ 1, 2, $\cdots$, n, then equation (35) is expressed as

$$(38) \quad R = \sum_{r=1}^{n} \lambda_r(\bar{x}_r\,q(\kappa))\,\bar{x}_r\,q(\kappa) = \sum_{r=1}^{n} R_r(\kappa).$$

PROPOSITION 2: *Let us rewrite equations (35) and (38) as a tax revenue function t of the multiplier κ as follows:*

$$(39) \quad t = t(\kappa) = \sum_{r=1}^{n} R_r(\kappa);\ \textit{a monotone decreasing as}\ \frac{d\,t}{d\,\kappa} < 0\ \textit{for}\ \kappa \leq 0,$$

then there exists a unique optimal solution κ^* at $\kappa = \kappa^* \leq 0$, corresponding to the government's constraint amount of R; $0 \leq R < \bar{R} \equiv \sum_{r=1}^{n} R_r^{max}$, a potential total maximum amount that she can gather from n-good markets.

Proof. See appendix A for this proof. □

B. *An Example with Only Affine Markets*

Consider a parametric example with n-good markets, in each of which a government faces not only an inverse demand function

$$(40) \quad \phi_r = \phi_r(x_r) \equiv -\alpha_r\,x_r + \beta_r \qquad \text{for } r = 1, 2, \cdots, n$$

but also a supply function

$$(41) \quad f_r = f_r(x_r) \equiv a_r\,x_r + b_r \qquad \text{for } r = 1, 2, \cdots, n$$

with parameters $\alpha_r > 0$, $\beta_r > b_r \geq 0$, and $a_r \geq 0$ but $a_r = b_r \neq 0$ at the same time, and she imposes an ad valorem tax rate μ_r on the r-th good in order to collect her tax revenue by an amount of R out of a total sum of the potentially maximal tax revenue $\hat{R} \equiv \sum_{r=1}^{n} \hat{R}_r^{max} = \sum_{r=1}^{n} \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$ as seen in equation (8).

COROLLARY 2: *As its tax revenue function t of the multiplier κ becomes*

$$(42)\quad t = t(\kappa) = 4\,\hat{R}\,\frac{\kappa\,(\kappa-1)}{(2\,\kappa-1)^2} \text{ and so that } \frac{d\,t}{d\,\kappa} = \frac{4\,\hat{R}}{(2\,\kappa-1)^3} < 0 \text{ for } \kappa \leq 0,$$

there exists a unique optimal solution κ^* as demonstrated in figure 3 of

$$(43)\quad \kappa^* = \frac{\hat{R} - R - \sqrt{(\hat{R}-R)\,\hat{R}}}{2\,(\hat{R}-R)} \leq 0$$

for her tax revenue R to be gathered out of the potential total maximum $\hat{R}$.

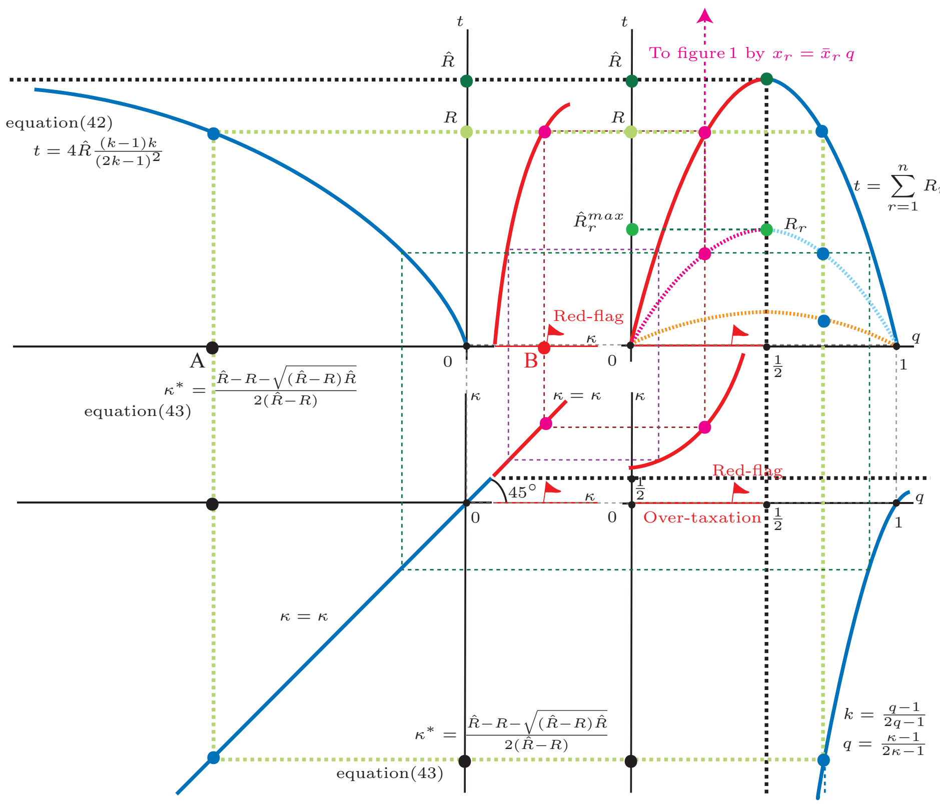

Figure 3: Corollary 2; optimal κ^* for R Attains at Point • A, not at • B

Then, an optimal ad valorem tax rate μ_r^* on the r-th good is computed as

$$(44)\quad \mu_r^* = \frac{(\alpha_r + a_r)\,(\beta_r - b_r)\,(\sqrt{\hat{R}} - \sqrt{\hat{R}-R})}{2\,(\alpha_r + a_r)\,b_r\,\sqrt{\hat{R}} + a_r\,(\beta_r - b_r)\,(\sqrt{\hat{R}} + \sqrt{\hat{R}-R})}$$

for her to obtain an optimal tax revenue R_r^* of

$$(45)\quad R_r^* = R\,\frac{\hat{R}_r^{max}}{\hat{R}} \text{ where } \hat{R} \equiv \sum_{r=1}^{n} \hat{R}_r^{max} \text{ and } \hat{R}_r^{max} \equiv \frac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$$

in the r-th market with $0 \leq R < \hat{R} \equiv \dfrac{1}{4}\displaystyle\sum_{r=1}^{n}\dfrac{(\beta_r - b_r)^2}{\alpha_r + a_r}$, *the total maximum.*

Proof. Even though the above figure 3 that reflects what is going on is likely to give us a pictorial proof, see appendixes A and B✍❶❷❸ in details. □

It is interesting for us to observe in equation (44) directly derived from equation (4) that every optimal ad valorem tax rate μ_r^* seems so complicated that no one is now able to insist on the Ramsey tax rules any more as stated in proposition 1 and its corollary 1; but in equation (45) that its complicated rate μ_r^* allows her to have a certain revenue R simply at a weighted portion $\dfrac{\hat{R}_r^{max}}{\hat{R}}$, the r-th potential maximum $\hat{R}_r^{max} = \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$ to their total sum $\hat{R}$. So, we can see that there do exist no trade-off between efficiency and equity because she just acquires the revenue R from among n markets in descending order of the biggest amount of the maximal $\hat{R}_r^{max}$ that the r-th market can potentially pay for an indirect tax. We can also see from equations (36) and (37) that at the optimal tax rate μ_r^* for all $r = 1, 2, \cdots$, and n, the optimal tax revenue R_r^* in equation (45) goes up 1 % causing equilibrium quantity x_r^* in every r-th market to reduce $|\kappa^*|$% identically as equation (43).[12]

C. *An Example with Hyperbolic Demand Functions*

Consider another one with n-good markets where a government meets a non-linear inverse demand function of

$$(46)\quad \phi_r = \phi_r(x_r) \equiv \frac{\gamma_r}{x_r}, \gamma_r > 0 \qquad \text{for } r = 1, 2, \cdots, n$$

whose price elasticity ρ_r of demand takes a value of unity everywhere in its domain of equation (2), say $\rho_r^{**} \equiv -\dfrac{\phi_r(x_r)}{\phi_r'(x_r)\,x_r} = 1$. Meanwhile, she has a linear supply function in each market as

$$(47)\quad f_r = f_r(x_r) \equiv a_r\, x_r, a_r > 0 \qquad \text{for } r = 1, 2, \cdots, n$$

[12] Instead of equation (44), *ceteris paribus*, an optimal unit tax υ_r^* for the r-th good in those homogeneous affine n markets is computed as $\upsilon_r^* = \frac{\beta_r - b_r}{2}\,\frac{\hat{R} - \sqrt{(\hat{R} - R)\,\hat{R}}}{\hat{R}}$ from footnote 5 or a numerator of equation (A16) with equation (A15) in order to collect an amount of R_r^* in equation (45) as her tax revenue for the r-th good market.

whose price elasticity ε_r of supply also takes a value of unity anywhere in its domain of equation (2), or $\varepsilon_r^{**} \equiv \dfrac{f_r(x_r)}{f_r'(x_r)\,x_r} = 1$. As discussed in section II.D, the sum function σ_r in equation (29) of reciprocals of those price elasticities takes a value of two as $\sigma_r^{**} \equiv \dfrac{1}{\rho_r^{**}} + \dfrac{1}{\varepsilon_r^{**}} = 2$ for all $r = 1, 2, \cdots$, and n.

In this case, she levies an ad valorem tax rate μ_r on the r-th good in order to collect her tax revenue by an amount of R out of the maximal tax revenue $\grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} = \sum_{r=1}^{n} \gamma_r$ as shown in equation (10).

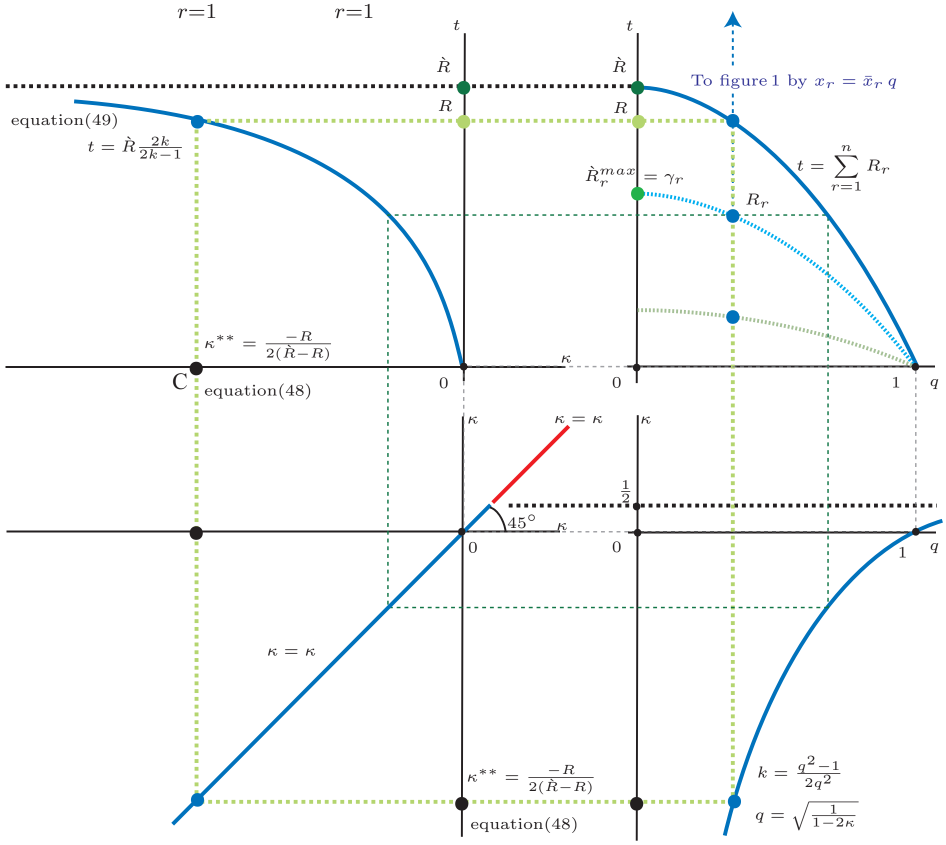

Figure 4: Corollary 3; optimal κ^{**} for R Holds at Point • C

COROLLARY 3: *There exists a unique optimal solution* κ^{**} *of*

$$(48)\quad \kappa^{**} = \frac{-R}{2\,(\grave{R} - R)} \leq 0 \text{ for her tax revenue } R;\, 0 \leq R < \grave{R} \equiv \sum_{r=1}^{n} \gamma_r$$

because its tax revenue function t *of the multiplier* κ *becomes*

$$(49)\quad t = t(\kappa) = 2\,\grave{R}\,\frac{\kappa}{2\,\kappa - 1} \text{ and so that } \frac{d\,t}{d\,\kappa} = \frac{-2\,\grave{R}}{(2\,\kappa - 1)^2} < 0 \text{ for } \kappa \leq 0.$$

So, an optimal ad valorem tax rate μ_r^{**} on the r-th good is computed as

$$(50)\quad \mu_r^{**} = \frac{R}{\grave{R} - R},\ 0 \leq R < \grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} \equiv \sum_{r=1}^{n} \gamma_r$$

to have an optimal tax revenue R_r^{**} of

$$(51)\quad R_r^{**} = R\,\frac{\grave{R}_r^{max}}{\grave{R}}\ \textit{where}\ \grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max}\ \textit{and}\ \grave{R}_r^{max} \equiv \gamma_r$$

from the r-th market for all $r = 1, 2, \cdots,$ and n.[13]

Proof. See appendixes A and B✍❹❺❻ in details with the above figure 4. □

Other things being equal, suppose only equation (47) were replaced by the following horizontal supply function of

$$(52)\quad f_r(x_r) \equiv b_r > 0 \qquad \text{for } r = 1, 2, \cdots, n$$

in corollary 3, then its price elasticity ε_r of supply diverges to infinity (∞) in its domain of equation (2), or $\bar{\varepsilon}_r^{**} \equiv \dfrac{f_r(x_r)}{f_r'(x_r)\,x_r} \to \infty$ letting its reciprocal $\dfrac{1}{\bar{\varepsilon}_r^{**}}$ converged to zero, and so making the sum function σ_r in equation (29) taken a value of unity as $\bar{\sigma}_r^{**} \equiv \dfrac{1}{\rho_r^{**}} + \dfrac{1}{\bar{\varepsilon}_r^{**}} = 1$ for all $r = 1, 2, \cdots,$ and n.

COROLLARY 4: *The government has a monotone decreasing tax function of $t = t(\kappa) = \grave{R}\,\dfrac{\kappa}{\kappa - 1}$ differently from equation (49) as well as an optimal solution $\bar{\kappa}^{**}$ of $\bar{\kappa}^{**} = \dfrac{-R}{\grave{R} - R} \leq 0$ rather than equation (48); however, she has exactly the same optimal ad valorem tax rate $\bar{\mu}_r^{**}$ as in equation (50):*[14]

$$(53)\quad \bar{\mu}_r^{**} = \frac{R}{\grave{R} - R},\ 0 \leq R < \grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} \equiv \sum_{r=1}^{n} \gamma_r$$

on the r-th good for its market's tax revenue $\bar{R}_r^{**} \equiv R\,\dfrac{\gamma_r}{\grave{R}}$ as equation (51).

Proof. See appendixes A and B✍❼❽❾ for this proof. □

[13] For this example, an optimal unit tax υ_r^{**} is given as $\upsilon_r^{**} = \sqrt{\gamma_r\, a_r}\, R \,/ \sqrt{(\grave{R} - R)\,\grave{R}}$ according to footnote 5 or a numerator of equation (A27) with equation (A25).

[14] With horizontal supply functions of equation (52), from footnote 5 or a numerator of equation (A28) with equation (A26), an optimal unit tax $\bar{\upsilon}_r^{**}$ turns to be $\bar{\upsilon}_r^{**} = b_r\, R \,/\, \grave{R}$.

It is interesting to see here in equations (50) and (53) that she has uniform optimal ad valorem tax rates μ_r^{**} as well as $\bar{\mu}_r^{**}$, *i.e.*, these rates do not seem to be influenced by supply functions such as equations (47) and (52) because their biggest maximal $\grave{R}_r^{max}$ in equation (51), which the r-th market can potentially pay for indirect taxes, consists of a parameter γ_r only in a demand equation (46). It is also interesting to see in equations (50) and (53) with equation (29) that no matter how the sum function σ_r, the reciprocal of price elasticities of demand and supply, might take a value of $\sigma_r^{**} = 2$ or $\bar{\sigma}_r^{**} = 1$, a ratio of tax rates from among μ_r^{**} or $\bar{\mu}_r^{**}$ becomes unity as

$$(54) \quad \frac{\mu_i^{**}}{\mu_j^{**}} = \frac{\bar{\mu}_i^{**}}{\bar{\mu}_j^{**}} = \frac{R}{\grave{R} - R} \, \frac{\grave{R} - R}{R} = 1$$

for all i, j, and $r \in \{r \mid r = 1, 2, \cdots, n\}$: That is, equation (54) immediately tells us that the Ramsey tax ratio rule discussed in corollary 1 malfunctions here again, and so that there is no trade-off between efficiency and equity. Then, based upon equation (51) with equation (50) or (53), she should collect her indirect tax R in descending order of the biggest amount of the potential maximum $\grave{R}_r^{max} = \gamma_r$ from among n markets for $r \in \{r \mid r = 1, 2, \cdots, n\}$.

D. *An Example with Mixed Markets*

Previously on a few of examples, all n markets are kinds of homogeneous where functions for demand or supply are supposed to take identical forms but these parameters' values are supposed to be different from each other. Let us consider here heterogeneous three-good markets where a government faces three inverse demand functions as

$$(40) \quad \phi_r = \phi_r(x_r) \equiv -\alpha_r \, x_r + \beta_r \qquad \text{for } r = 1$$

$$(46) \quad \phi_r = \phi_r(x_r) \equiv \frac{\gamma_r}{x_r} \qquad \text{for } r = 2, 3$$

as well as the following three supply functions of

$$(41) \quad f_r = f_r(x_r) \equiv a_r \, x_r + b_r \qquad \text{for } r = 1$$

$$(47) \quad f_r = f_r(x_r) \equiv a_r \, x_r \qquad \text{for } r = 2, 3$$

with parameters $\alpha_r > 0$, $\beta_r > b_r \geq 0$, $\gamma_r > 0$, and $a_r \geq 0$ but $a_r = b_r \neq 0$ simultaneously, and she imposes an ad valorem tax rate μ_r on the r-th good in order to collect her tax revenue by an amount of R out of the potentially maximal tax revenue $\hat{R}^{max}_{mix} \equiv \hat{R}^{max}_1 + \grave{R}^{max}_2 + \grave{R}^{max}_3$ where $\hat{R}^{max}_1 \equiv \dfrac{(\beta_1 - b_1)^2}{4\,(\alpha_1 + a_1)}$ and $\grave{R}^{max}_r \equiv \gamma_r$ for $r = 2$ and 3 as shown in equations (8) and (10), respectively.

COROLLARY 5: *As its tax revenue function t of the multiplier κ becomes*

$$(55)\quad t = t(\kappa) = 4\,\hat{R}^{max}_1 \frac{\kappa\,(\kappa - 1)}{(2\,\kappa - 1)^2} + 2\,\gamma_2 \frac{\kappa}{2\,\kappa - 1} + 2\,\gamma_3 \frac{\kappa}{2\,\kappa - 1}$$

in this example, then there exists a unique optimal solution $\kappa^*_{mix} \leq 0$ of

$$(56)\quad \kappa^*_{mix} = \frac{2\,(\hat{R}^{max}_{mix} - R) - (\gamma_2 + \gamma_3) - \sqrt{4\,(\hat{R}^{max}_{mix} - R)\,\hat{R}^{max}_1 + (\gamma_2 + \gamma_3)^2}}{4\,(\hat{R}^{max}_{mix} - R)}$$

for her tax revenue R; $0 \leq R < \hat{R}^{max}_{mix} = \dfrac{1}{4}\,\dfrac{(\beta_1 - b_1)^2}{\alpha_1 + a_1} + \gamma_2 + \gamma_3$.

Proof. See appendix A for this proof. □

It is interesting to see in a quadratic equation (A29) of appendix A as

$$(A29)\quad 4\,(\hat{R}^{max}_{mix} - R)\,\kappa^2 - 2\,(\hat{R}^{max}_{mix} + \hat{R}^{max}_1 - 2\,R)\,\kappa - R = 0,$$

which is derived from equation (55) equal to R, the more heterogeneities of good markets increase, the higher order in a polynomial equation of κ shows up: For example, if we replace the last term $2\,\gamma_3 \dfrac{\kappa}{2\,\kappa - 1}$ in equation (55) by $\gamma_3 \dfrac{\kappa}{\kappa - 1}$ in corollary 4 or equation (A24), we must solve a cubic equation of

$$4\,\hat{R}^{max}_1 \frac{\kappa\,(\kappa - 1)}{(2\,\kappa - 1)^2} + 2\,\gamma_2 \frac{\kappa}{2\,\kappa - 1} + \gamma_3 \frac{\kappa}{\kappa - 1} = R;$$

perhaps without a parametrically trim root nor an algebraic solution by the factor theorem as the higher order a polynomial equation has, algebraically speaking, the less opportunity to use the general formula of roots we have.

Let us have a numerical example to ease in us for calculations, namely, not merely to make each potential maximum tax revenue taken a value of

$\hat{R}_1^{max} = 2$, $\grave{R}_2^{max} = 1$, and $\grave{R}_3^{max} = 3$ with parameters $a_3 = \dfrac{1}{3}$, $\alpha_1 = \gamma_2 = a_1 = b_1 = 1$, $\gamma_3 = 3$, $a_2 = 4$, and $\beta_1 = 5$, but also to make her collect R as her tax revenue by an amount $R = 3.5$ out of a total maximum amount $\hat{R}_{mix}^{max} = \dfrac{1}{4}\dfrac{(\beta_1 - b_1)^2}{\alpha_1 + a_1} + \gamma_2 + \gamma_3 = 6$. Then, each market's initial equilibrium quantity $\bar{x}_r$ before tax, an upper limit of the domain in equation (2) for $r = 1, 2$, and 3, is calculated as $\bar{x}_1 = 2$, $\bar{x}_2 = 0.5$, and $\bar{x}_3 = 3$, respectively, by equations (40), (41), (46), and (47). From equation (56), she can numerically determine an optimal value of the multiplier denoted by $\tilde{\kappa}_{mix}^*$ as

$$(57)\quad \tilde{\kappa}_{mix}^* \equiv \frac{2\,(6-3.5)-(1+3)-\sqrt{4\,(6-3.5)\,2+(1+3)^2}}{4\,(6-3.5)} = \frac{1-6}{10}.$$

Substitute this equation (57) or $\tilde{\kappa}_{mix}^* = -0.5$ into equations (A11) and (A21) to obtain unit-less quantities qs in equation (A5) as $q_1 = \dfrac{3}{4}$ for market 1 and as $q_2 = q_3 = \dfrac{\sqrt{2}}{2}$ for markets 2 and 3, then we have equilibrium quantities after tax as $\tilde{x}_1^* = \bar{x}_1\, q_1 = 1.5$, $\tilde{x}_2^* = \bar{x}_2\, q_2 = \dfrac{\sqrt{2}}{4}$, and $\tilde{x}_3^* = \bar{x}_3\, q_3 = \dfrac{3\sqrt{2}}{2}$. So, equations (A16) and (A27) provide us with optimal ad valorem tax rates $\tilde{\mu}_1^* = 0.4$ as 40% and $\tilde{\mu}_2^* = \tilde{\mu}_3^* = 1$ as 100%, respectively; whereas according to footnote 5, numerators of equations (A16) and (A27) provide us with not only optimal unit taxes but also vertical lengths as $\tilde{v}_1^* = \tilde{\lambda}_1^* = 1$, $\tilde{v}_2^* = \tilde{\lambda}_2^* = \sqrt{2}$, and $\tilde{v}_3^* = \tilde{\lambda}_3^* = \dfrac{\sqrt{2}}{2}$. Now, Equation (6) tells us that she is able to collect her tax revenue $R = 3.5$ as $\tilde{R}_1^* = \tilde{\lambda}_1^*\,\tilde{x}_1^* = 1.5$, $\tilde{R}_2^* = \tilde{\lambda}_2^*\,\tilde{x}_2^* = 1$, and $\tilde{R}_3^* = \tilde{\lambda}_3^*\,\tilde{x}_3^* = 1.5$ whose results, collected taxes $\tilde{R}_r^*$ increase 1% causing equilibrium quantities $\tilde{x}_r^*$ to fall 0.5% equally for $r = 1, 2$, and 3 from equation (37), can be checked out by substituting $\tilde{\kappa}_{mix}^* = -0.5$ of equation (57) into each term of equation (55): *i.e.*, $\tilde{R}_1^* = 4\,\hat{R}_1^{max}\dfrac{\tilde{\kappa}_{mix}^*\,(\tilde{\kappa}_{mix}^* - 1)}{(2\,\tilde{\kappa}_{mix}^* - 1)^2}$, $\tilde{R}_2^* = 2\,\gamma_2\,\dfrac{\tilde{\kappa}_{mix}^*}{2\,\tilde{\kappa}_{mix}^* - 1}$, and $\tilde{R}_3^* = 2\,\gamma_3\,\dfrac{\tilde{\kappa}_{mix}^*}{2\,\tilde{\kappa}_{mix}^* - 1}$. Finally, the following bordered principal minors:

$$|\bar{H}_2| \equiv \begin{vmatrix} 0 & -2 & -2\sqrt{2} \\ -2 & -4 & 0 \\ -2\sqrt{2} & 0 & -16 \end{vmatrix}\ ;\ |\bar{H}_3| \equiv \begin{vmatrix} 0 & -2 & -2\sqrt{2} & -\sqrt{2} \\ -2 & -4 & 0 & 0 \\ -2\sqrt{2} & 0 & -16 & 0 \\ -\sqrt{2} & 0 & 0 & -4/3 \end{vmatrix}$$

are satisfied with the second-order condition due to our appendix B ✌①②③.

Ⅲ. Concluding Remarks

In this paper, we have mainly had two purposes: One is a reexamination of the "*Ramsey rules*" or "*his reciprocal elasticity rule* and its implication as a rule that high taxes should be levied on commodities inelastic demand;" the other is seeking some parametric closed-form solutions that no one has yet shown before in the literature because Frank P. Ramsey (1927) @p.56 did show us *reciprocal-elasticity*-part in *his equation* (11), but he did not show us a mathematical proof on its *rule*-part, just verbally stating in *his equation* (12) as "infinitesimal $\cdots$ tax *ad valorem* on each commodity should be proportional to the sum of the reciprocals of its supply and demand elasticities," and because since then, he and his followers have used what they should have proved and so made their rules from such an "infinitesimal" tax, believing without a closed-form solution that these rules could be perfectly valid for "a tax of 500% on whisky," *ibid.*, @p.60.

To achieve our purposes, we have used exactly the same model as Frank P. Ramsey (1927) @p.55-8. As seen in proposition 1 and appendixes A and C, the limit of the $\frac{0}{0}$ form known as the l'Hôpital's rule as well as *Infinitesimal Calculus* reveal that the "*Ramsey rule*" or "*reciprocal elasticity rule*" was his verbal illusion: That is, there does not exist anywhere such a proportionality between an "infinitesimal" indirect tax and a sum of reciprocals of price elasticities of supply ε_r and demand ρ_r even with a horizontal supply function because the sum function $\sigma_r = \frac{1}{\varepsilon_r} + \frac{1}{\rho_r}$ in equation (29) is always too large to have the same order in the $\frac{0}{0}$ form as "infinitesimal" candidates of six equations (22) through (26), and (C3) with its limit; then the limit of the $\frac{0/0}{0/0}$ form with the Leibniz's notation shows us in corollary 1 and appendix A that any ratio of optimal taxes such as $\frac{\mu_i^*}{\mu_j^*}$ ad valorem with an "infinitesimal" percentage-reduction K^* or $d\,K^*$ no longer implicates in an accurate one but an approximation to its derivative $\frac{d\,\mu_i^*}{d\,\mu_j^*} = \frac{d\,\mu_i^*}{d\,K^*}\,\frac{d\,K^*}{d\,\mu_j^*}$ at most. If its accurate one is needed, our equation (4) is go for it. As it contains no price elasticities reciprocally nor explicitly, there is no trade-off between efficiency and equity.

Although we have exactly the identical model-setting as Frank P. Ramsey (1927) @p.55-8, we obtain $(n + 1)$ equations (16) or (C5) and (17) for the first-order condition with a Lagrange multiplier κ $(= -K$, *ibid.*, @p.50), but he has n equations (C3) for it with another multiplier $-\theta$, *ibid.*, @p.55. As seen in appendix C, a difference between equations (C5) and (C3) is their denominators' λ_r, the length in equation (1) or (3) as $K \equiv \dfrac{-\lambda_r}{\lambda_r + \lambda'_r\, x_r}$ and $\theta_r \equiv \dfrac{-\lambda_r}{\lambda'_r\, x_r}$. Now, it is interesting to see that his equal-percentage-reduction rule, say $\theta = \theta_r$ is nothing more than the length λ_r elasticity of width x_r, an equilibrium quantity after tax, defined as equation (C3) with a rectangle R_r in figure 1 that reflects a tax revenue R_r in equation (6), whereas our equal-percentage-reduction rule $|\kappa| = |-K| = K$ in equation (16), (C5), (36), or (37) works for its area very as the tax revenue R_r elasticity of equilibrium quantity x_r after tax, or R_r increases 1 % causing x_r to reduce $|\kappa|$%. So, our rule K is a little bit different from his θ by $\dfrac{\theta^2}{1-\theta}$ as much as a distance $K - \theta$ because of $K = \dfrac{\theta}{1-\theta}$ in equation (C6) or (C7). Thus, it is also interesting to see with equations (C5) and (C3) again that this distance could not shrink to nil till the length λ_r in both numerators has become null, or until, in other words, the length λ_r has been removed out of n denominators in K, equation (C5).

As found it in the above paragraph that lacks of λ_r in n denominators of the Ramsey's first-order condition must result in the direct reason why and how his equal-percentage-reduction rule θ malfunctions, any modification on the Ramsey's assumptions such as a constant marginal utility of money, an income effect, a quadratic utility function, a horizontal supply function, and *etc.*, as his follows usually do may have nothing to do with its malfunctioning. Then, such a modification might be done for nothing because the length λ_r consists of an inverse demand function ϕ_r and a supply function f_r as $\lambda_r = \phi_r - f_r$ in equation (1) or (3), and so because if any, we should have modified both functions' effects and dealt with their offsets at the same time.

As pointed out in our introductory part and shortly after equation (C6), the reason why Frank P. Ramsey (1927) @p.55 has n equations for the first-order condition and how he throws away the multiplier $-K$ to take his rule θ seems to be a tradition as Paul A. Samuelson (1951) @p.87 puts it on that "we can eliminate the Lagrangian multiplier $\cdots$, and write down the n relations determining the n unknowns, $(t_1, \cdots, t_n)$, as (8.7). $\cdots$ Ramsey's problem is formally solved by (8.7). However, the result is not very intuitive." Against the tradition, as developed in section II.A, we have determined the $(n+1)$-th unknown of our multiplier $\kappa = -K$ in the first place as an elimination of it first and foremost must often leave the first-order condition unfinished as a necessary condition for an optimality of indirect taxes in the model.

As obtained in corollaries 2 through 5, we have several parametric closed-form solutions for optimal ad valorem tax rates μ_r as well as unit taxes υ_r in affine markets*, hyperbolic demand markets**, and mixed markets$^{\tilde{}*}$ as

$$(44) \quad \mu_r^* = \frac{(\alpha_r + a_r)\,(\beta_r - b_r)\,(\sqrt{\hat{R}} - \sqrt{\hat{R} - R})}{2\,(\alpha_r + a_r)\,b_r\,\sqrt{\hat{R}} + a_r\,(\beta_r - b_r)\,(\sqrt{\hat{R}} + \sqrt{\hat{R} - R})},$$

$$(\text{f12}) \quad \upsilon_r^* = (\beta_r - b_r)\left\{\hat{R} - \sqrt{(\hat{R} - R)\,\hat{R}}\right\}/(2\,\hat{R}), \text{ and } \hat{R} \equiv \sum_{r=1}^{n} \frac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)},$$

$$(50) \quad \mu_r^{**} = R/(\grave{R} - R),$$

$$(\text{f13}) \quad \upsilon_r^{**} = \sqrt{\gamma_r\,a_r}\,R/\sqrt{(\grave{R} - R)\,\grave{R}}, \text{ and } \grave{R} \equiv \sum_{r=1}^{n} \gamma_r,$$

$$(53) \quad \bar{\mu}_r^{**} = R/(\grave{R} - R),$$

$$(\text{f14}) \quad \bar{\upsilon}_r^{**} = b_r\,R/\grave{R}, \text{ and } \grave{R} \equiv \sum_{r=1}^{n} \gamma_r,$$

$$(\text{c}\ddot{\text{v}}) \quad \tilde{\mu}_1^* = 40\%, \ \tilde{\mu}_2^* = \tilde{\mu}_3^* = 100\%, \ \tilde{\upsilon}_1^* = 1, \ \tilde{\upsilon}_2^* = \sqrt{2}, \text{ and } \tilde{\upsilon}_3^* = \sqrt{2}/2,$$

in which f in equation number # or (#) stands for footnote and cv̈ in that for corollary 5, and parameters are given nearby equations (#), respectively. As Frank P. Ramsey (1927) @p.52 assumes that "λ's are linear" for the length λ_r in equation (1) or (3), he should have solved his problem (*ibid.*, @p.55 put it into equation (15)) at least as equation (44) for affine, linear plus intercepts, markets. Frank P. Ramsey (1927) @p.60 conjectures, as cited here and there, "the more complicated results $\cdots$ may well be valid under still wider conditions" than his assumption that "λ's are linear," *ibid.*, @p.52. However, ours with non-linear lengths λ_r in corollaries 3 through 5 seem to be simpler than those with linear ones in equations (44) and (f12). Needless to say, our optimal solutions are always valid everywhere in equation (2) of their entire domains except an over-taxation area with red-flags drawn in figures 1 and 3.

As proved in appendix B by the mathematical induction, we have shown the second-order condition for equation (15), due to Frank P. Ramsey (1927) @p.55, with a bordered Hessian $|\bar{H}|$. In addition to this sufficiency with $|\bar{H}|$, equation (B1), a monotone decreasing as $\dfrac{d\,t}{d\,\kappa} < 0$ strictly upon a tax revenue function t of our multiplier κ or $t = t(\kappa)$ in equation (39) of proposition 2 has provided us with a necessary and sufficient condition for a unique optimal solution of an indirect taxation, which can automatically stay a government away from the red-flag over-taxation area in figures 1 and 3. We have had in the sufficient condition that not merely a slope but also a slope of its slope or a curvature of $R_r = R_r(x_r)$, an each market's tax revenue function R_r of an equilibrium quantity x_r after tax in equation (6), has an important role on it such as a negative slope of $\dfrac{d\,R_r}{d\,x_r} \equiv R_r' < 0$ in equation (13) and a quasi-concavity of $\dfrac{d^2 R_r}{d\,x_r^2} \equiv R_r'' \leq 0$ in equation (12), so that in this way, it deserves studying if "results $\cdots$ may well be valid under still wider conditions" as Frank P. Ramsey (1927) @p.60 guesses: *e. g.*, diagonal elements $\mathcal{L}_{rr}$ in equation (B1) should be negative for all $r = 1, 2, \cdots, n$ as $\mathcal{L}_{rr} \equiv \lambda_r' - \kappa\, R_r'' < 0$ in equation (B4) with $\lambda_r' < 0$ and $\kappa \leq 0$ in equations (8) and (A7), but it seems possible for $\mathcal{L}_{rr}$ to be negative "under still wider conditions;" how about off-diagonal elements $\mathcal{L}_{rs}$ for $r \neq s$? although we leave them open to question.

Since 1927, Frank P. Ramsey and his followers have used what they should have proved on an "infinitesimal" tax and believed without a closed-form solution that their rules could work even for "a tax of 500% on whisky." Now, we believe that optimal rules, solutions, and methods developed here seem to be so tractable and useful that we can easily improve them as well as apply them further to international tariffs, incomplete markets with economic externalities, and anything even to a labor income tax, one of direct taxes.

Appendix

A. *Proofs*

In this appendix, we prove our propositions and their corollaries if any.

☛ POOF OF PROPOSITION 1:

Proof. Suppose not, or the Ramsey tax rule were not an illusion: *i. e.*, it were true that we had a proportionality of $\mu_r \propto \sigma_r$ for an infinitesimal μ_r. Then, the limit of $\sigma_r = \sigma_r(x_r)$ in equation (29) must be convergent to zero as x_r goes to an initial equilibrium quantity of $\bar{x}_r$ in the set of $0 < x_r < \bar{x}_r$:

$$\text{(A1)} \quad \lim_{x_r \to \bar{x}_r^-} \sigma_r = \frac{1}{\rho_r} + \frac{1}{\varepsilon_r} = -\frac{\phi_r'(\bar{x}_r)\,\bar{x}_r}{\phi_r(\bar{x}_r)} + \frac{f_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} = 0.$$

At the initial quantity of $\bar{x}_r > 0$ before tax, one has $\phi_r(\bar{x}_r) = f_r(\bar{x}_r)$, demand = supply given in equation (2). Thus, equation (A1) is reduced into

$$\frac{f_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} - \frac{\phi_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} = \{f_r'(\bar{x}_r) - \phi_r'(\bar{x}_r)\}\frac{\bar{x}_r}{f_r(\bar{x}_r)} = 0, \text{ or}$$

$$\text{(A2)} \quad f_r'(\bar{x}_r) = \phi_r'(\bar{x}_r),$$

which contradicts the signs of slopes: $0 \leq f_r'(\bar{x}_r) = \phi_r'(\bar{x}_r) < 0$. So, we have

$$\text{(A3)} \quad \lim_{x_r \to \bar{x}_r^-} \sigma_r(x_r) = \sigma_r(\bar{x}_r) = -\frac{\{\phi_r'(\bar{x}_r) - f_r'(\bar{x}_r)\}\,\bar{x}_r}{f_r(\bar{x}_r)} = -\frac{\lambda_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} > 0$$

instead of equation (A1) by means of equation (8) or $\lambda_r' < 0$. Differently from five equations (22) through (26), the sum function $\sigma_r(x_r)$ in equation (29) is too large even at $x_r = \bar{x}_r$ to have the same order as these five equations, so that we cannot obtain the $\frac{0}{0}$ form against the function $\sigma_r = \sigma_r(x_r)$ forever.

On the other hand, it is interesting to see from the following limit of the $\frac{0}{0}$ form of equations (24) to (22) as x_r runs to $\bar{x}_r$ from the left hand side that

$$\text{(A4)} \quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{K_r} = \frac{\lambda_r(\bar{x}_r)}{f_r(\bar{x}_r)} \, \frac{[\{\lambda_r(x_r)\}] + \lambda_r'(\bar{x}_r)\,\bar{x}_r}{-\lambda_r(\bar{x}_r)} = \sigma_r(\bar{x}_r) \equiv c_r,$$

in a numerator of which a term denoted by $[\{\lambda_r(x_r)\}]$ of the vertical length in equation (1) or (3) should be always null as $[\{\lambda_r(x_r)\}] \approx 0$ at $x_r < \bar{x}_r$ for an infinitesimal rate μ_r due to equations (23) and (24). Thus, an ad valorem tax rate μ_r can be proportional to ($\propto$) a percentage-reduction K_r or $\mu_r \propto K_r$ with a straight line of $\mu_r = c_r\,K_r$ nearby at $x_r \leq \bar{x}_r$ for an infinitesimal tax rate $\mu_r \geq 0$ where a constant coefficient c_r is given in equation (A4). □

☞ POOF OF COROLLARY 1:

Proof. As each quantity x_r may be measured by a different unit, we translate its domain of $0 < x_r \leq \bar{x}_r$ in equation (2) into a common q of

$$\text{(A5)}\quad 0 < q = q_r \equiv \frac{x_r}{\bar{x}_r} \leq 1.$$

Then, equation (A4) is able to provide us with the limit of a ratio of

$$\text{(A6)}\quad \lim_{q \to 1^-} \frac{\mu_i / K_i}{\mu_j / K_j} = \lim_{q \to 1^-} \frac{\sigma_i(\bar{x}_i\, q)}{\sigma_j(\bar{x}_j\, q)} = \frac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)} \equiv c_{ij},$$

which immediately tells us even in the neighborhood of $x_r = \bar{x}_r$ or $q = 1$ that $\frac{\mu_i}{\mu_j} \not\propto \frac{\sigma_i(x_i)}{\sigma_j(x_j)}$ but $\frac{\mu_i}{\mu_j} \propto \frac{K_i(x_i)}{K_j(x_j)}$ at $x_r \leq \bar{x}_r$ but almost near at $\bar{x}_r$ having a linear relationship as $\frac{\mu_i(\bar{x}_i\, q)}{\mu_j(\bar{x}_j\, q)} = c_{ij}\, \frac{K_i(\bar{x}_i\, q)}{K_j(\bar{x}_j\, q)}$ with a constant slope of $c_{ij} \equiv \frac{\sigma_i(\bar{x}_i)}{\sigma_j(\bar{x}_j)}$ from equation (A6) nearby at $q \leq q_r = 1$ for all i, j, and $r \in \{r \mid r = 1, 2, \cdots, n\}$. □

☛ POOF OF PROPOSITION 2:

Proof. First of all, the function t of κ, $t = t(\kappa)$ of equation (39), has a domain of $\kappa \leq 0$ that is derived from equations (16), (36), and (37) as

$$\text{(A7)}\quad \kappa = \frac{\lambda_r}{\lambda_r + \lambda_r'\, x_r} = \frac{R_r / x_r}{d\, R_r / d\, x_r} = \frac{R_r / q}{d\, R_r / d\, q} \leq 0$$

due to a non-negative numerator of the vertical length $\lambda_r \geq 0$ in equation (1) or (3), and due to a negative denominator of the marginal tax revenue $\frac{d\, R_r}{d\, x_r} < 0$ in equation (13). It is trivial for all $r = 1, 2, \cdots, n$ that $\lambda_r = \kappa = t = 0$. Next, for a certain amount of $R_r^{\exists}$ less than the r-th market's maximum $R_r^{max} : 0 \leq R_r^{\exists} < R_r^{max}$; in total, we have $0 \leq R \equiv \sum_{r=1}^{n} R_r^{\exists} < \bar{R} \equiv \sum_{r=1}^{n} R_r^{max}$. Then, equation (39), the function $t = t(\kappa)$, has a region of $0 \leq t = R < \bar{R}$. By employing the chain rule, at last, it is easy to see in the domain $\kappa \leq 0$ that the tax revenue function $t = t(\kappa)$ is monotone decreasing with respect to the multiplier κ as

$$\text{(A8)}\quad \frac{d\, t}{d\, \kappa} = \sum_{r=1}^{n} \frac{d\, R_r}{d\, x_r} \frac{d\, x_r}{d\, \kappa} = \sum_{r=1}^{n} \frac{d\, R_r}{d\, x_r} \frac{1}{d\, \kappa / d\, x_r} < 0$$

because of not only the marginal tax revenue of equation (13) as $\dfrac{d\,R_r}{d\,x_r} < 0$ but also the following marginal multiplier of equation (16) as $\dfrac{d\,\kappa}{d\,x_r} > 0$, or

$$\begin{aligned}\frac{d\,\kappa}{d\,x_r} &= \frac{\lambda_r'\,(\lambda_r + \lambda_r'\,x_r) - \lambda_r\,(2\,\lambda_r' + \lambda_r''\,x_r)}{(\lambda_r + \lambda_r'\,x_r)^2} \\ &= \frac{\lambda_r'\,(d\,R_r/d\,x_r) - \lambda_r\,(d^2\,R_r/d\,x_r^2)}{(d\,R_r/d\,x_r)^2} > 0\end{aligned}$$

owing to equations (1) or (3), (8), (12), and (13) as $\lambda_r \geq 0$, $\lambda_r' < 0$, $\dfrac{d^2\,R_r}{d\,x_r^2} \leq 0$, and $\dfrac{d\,R_r}{d\,x_r} < 0$, respectively, some of which simultaneously make sure of the second-order condition according to equations (B3) and (B4) in appendix B. Consequently, such a monotone decreasing $\dfrac{d\,t}{d\,\kappa} < 0$ guarantees us a unique optimal solution κ^* for $t = R$ somewhere in its domain $\kappa \leq 0$. □

☞ POOF OF COROLLARY 2:

Proof. As developed in section Ⅱ.A, we employ the third procedure of equation (37) or (A7) with the unit-less quantity q of equation (A5) as

$$\text{(37)} \quad \kappa = \frac{R_r/q}{d\,R_r/d\,q} \leq 0;\ \ 0 < q = q_r \equiv \frac{x_r}{\bar{x}_r} \leq 1 \quad \text{for } r = 1, 2, \cdots, n.$$

First of all, equalizing equations (40) and (41) of $-\,\alpha_r\,x_r + \beta_r = a_r\,x_r + b_r$ gives us each market's initial quantity $\bar{x}_r$ before tax as $\bar{x}_r^B \equiv \dfrac{\beta_r - b_r}{\alpha_r + a_r}$. Then, we can put not only a quantity variable x_r in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$ into equation (A5) as $x_r = \bar{x}_r^B\,q$ but also a tax revenue variable R_r of equation (6) or $R_r = (-\,\alpha_r\,x_r + \beta_r - a_r\,x_r - b_r)\,x_r$ in this case into

$$\text{(A9)} \quad R_r^B \equiv -\,4\,\hat{R}_r^{max}\,q^2 + 4\,\hat{R}_r^{max}\,q = -\,4\,\hat{R}_r^{max}\,q\,(q-1)$$

where $\hat{R}_r^{max}$ is a maximal amount in equation (8), which every market can potentially pay for an indirect tax; as $q \to 0.5^+$ in all markets, it approaches $\hat{R}_r^{max} = \dfrac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$. So, our equation (37) conveys equation (A9) into

$$\text{(A10)} \quad \kappa = \kappa(q) \equiv \frac{R_r^B/q}{d\,R_r^B/d\,q} = \frac{q-1}{2\,q-1} \leq 0;\ \ 0.5 < q = q_r \equiv \frac{x_r}{\bar{x}_r^B} \leq 1$$

due to an average function $\dfrac{R_r^B}{q} = -\,4\,\hat{R}_r^{max}\,(q-1)$ and the derivative $\dfrac{d\,R_r^B}{d\,q} = -\,4\,\hat{R}_r^{max}\,(2q-1)$. And its inverse function q is provided as

$$\text{(A11)} \quad q = q(\kappa) \equiv \frac{\kappa - 1}{2\,\kappa - 1};\ \ \kappa \leq 0.$$

By using equation (39) in proposition 2 or equation (38) for this proof, next, a sum of equation (A9) with equation (A11) for a tax revenue function t of κ equal to R, a government's constraint for her tax revenue, is given as

$$\text{(A12)}\quad t = t(\kappa) \equiv \sum_{r=1}^{n} -4\,\hat{R}_r^{max}\, q\,(q-1) = -4\,\hat{R}\,\frac{\kappa-1}{2\,\kappa-1}\,\frac{-\kappa}{2\,\kappa-1} = R$$

where $\hat{R}$ is the total sum as $\hat{R} \equiv \sum_{r=1}^{n} \hat{R}_r^{max} = \sum_{r=1}^{n} \frac{(\beta_r - b_r)^2}{4\,(\alpha_r + a_r)}$; $0 \leq R < \hat{R}$. Equation (A12) is nothing more than equation (42) itself, which yields the following quadratic function to us:

$$\text{(A13)}\quad (\hat{R}-R)\,\kappa^2 - (\hat{R}-R)\,\kappa - \frac{R}{4} = 0;\;\; \kappa \leq 0,\;\; 0 \leq R < \hat{R}.$$

Its discriminant, say D is always positive as $D \equiv (\hat{R}-R)^2 + (\hat{R}-R)\,R = (\hat{R}-R)\,\hat{R} > 0$ so that its quadratic formula produces two roots of

$$\text{(A14)}\quad \kappa = \frac{\hat{R} - R \pm \sqrt{(\hat{R}-R)\,\hat{R}}}{2\,(\hat{R}-R)}$$

whose negative root directly becomes equation (43) owing to the domain of $\kappa \leq 0$. Finally, substituting equation (43) back into equation (A11) as

$$\text{(A15)}\quad q = \frac{\kappa^* - 1}{2\,\kappa^* - 1} = \frac{\hat{R} - R + \sqrt{(\hat{R}-R)\,\hat{R}}}{2\sqrt{(\hat{R}-R)\,\hat{R}}} = \frac{\hat{R} + \sqrt{(\hat{R}-R)\,\hat{R}}}{2\,\hat{R}}$$

and plugging its substituted equation (A15) back into equation (A9) sequentially provides us with equation (45) immediately. On the other hand, plugging the above equation (A15) back into equation (4) with $x_r = \bar{x}_r^B\, q$ as

$$\text{(A16)}\quad \mu_r = \frac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \frac{-\alpha_r\,\bar{x}_r^B\, q + \beta_r - a_r\,\bar{x}_r^B\, q - b_r}{a_r\,\bar{x}_r^B\, q + b_r},$$

one has equation (44) at once where $\bar{x}_r^B$ is the initial quantity $\bar{x}_r$ before tax of $\bar{x}_r^B \equiv \frac{\beta_r - b_r}{\alpha_r + a_r}$ as discussed earlier in this corollary. □

☞ POOF OF COROLLARIES 3 and 4:

Proof. Similarly to the previous proof, first of all, setting equation (46) equal to equations (47) and (52) as $\frac{\gamma_r}{x_r} = a_r\, x_r$ and as $\frac{\gamma_r}{x_r} = b_r$, respectively, gives us each market's initial quantity $\bar{x}_r$ before tax of $\bar{x}_r^C \equiv \sqrt{\gamma_r / a_r}$ and of $\bar{x}_r^H \equiv \dfrac{\gamma_r}{b_r}$. Then, we can transfer not only a quantity variable x_r in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$ into equation (A5) as $x_r = \bar{x}_r^C\, q$ as well as $x_r = \bar{x}_r^H\, q$ but also a tax revenue variable R_r of equation (6) or $R_r^C \equiv (\dfrac{\gamma_r}{x_r} - a_r\, x_r)\, x_r$ and $R_r^H \equiv (\dfrac{\gamma_r}{x_r} - b_r)\, x_r$ here into

$$\text{(A17)}\quad R_r^C = \grave{R}_r^{max}\,(1 - q^2),$$

in which $\grave{R}_r^{max}$ is a maximal amount in equation (10) that each market can potentially pay for an indirect tax: *i. e.*, it arrives at $\grave{R}_r^{max} = \gamma_r$ as $q \to 0^+$ in every market; and into

$$\text{(A18)}\quad R_r^H = \grave{R}_r^{max}\,(1 - q),$$

respectively. So, our equation (37) puts equations (A17) and (A18) into

$$\text{(A19)}\quad \kappa = \kappa(q) \equiv \frac{R_r^C / q}{d\, R_r^C / d\, q} = \frac{q^2 - 1}{2\, q^2} \leq 0;\ 0 < q = q_r \equiv \frac{x_r}{\bar{x}_r^C} \leq 1$$

because of $\dfrac{R_r^C}{q} = \dfrac{\grave{R}_r^{max}\,(1 - q^2)}{q}$ as well as $\dfrac{d\, R_r^C}{d\, q} = -\ 2\, \grave{R}_r^{max}\, q$; and into

$$\text{(A20)}\quad \kappa = \kappa(q) \equiv \frac{R_r^H / q}{d\, R_r^H / d\, q} = \frac{q - 1}{q} \leq 0;\ 0 < q = q_r \equiv \frac{x_r}{\bar{x}_r^H} \leq 1$$

owing to $\dfrac{R_r^H}{q} = \dfrac{\grave{R}_r^{max}\,(1 - q)}{q}$ and the derivative $\dfrac{d\, R_r^H}{d\, q} = -\ \grave{R}_r^{max}$. Then, their inverse functions for equations (A19) and (A20) are calculated as

$$\text{(A21)}\quad q = q(\kappa) \equiv \sqrt{\frac{1}{1 - 2\,\kappa}};\ \kappa \leq 0,$$

$$\text{(A22)}\quad q = q(\kappa) \equiv \frac{1}{1 - \kappa};\ \kappa \leq 0,$$

respectively. A sum of equation (A17) with equation (A21) for a function t of κ equal to R, a government's tax revenue, is given as equation (49) of

$$\text{(A23)}\quad t = t(\kappa) \equiv \sum_{r=1}^{n} \grave{R}_r^{max}\,(1 - q^2) = \grave{R}\,\frac{2\kappa}{2\,\kappa - 1} = R$$

where the total sum $\grave{R} \equiv \sum_{r=1}^{n} \grave{R}_r^{max} = \sum_{r=1}^{n} \gamma_r > R \geq 0$. In a similar manner,

$$\text{(A24)}\quad t = t(\kappa) \equiv \sum_{r=1}^{n} \grave{R}_r^{max}\,(1 - q) = \grave{R}\,\frac{\kappa}{\kappa - 1} = R$$

due to a sum of equation (A18) with equation (A22). Solving equation (A23) of $2\,\kappa\,\grave{R} = (2\,\kappa\,-\,1)\,R$ with respect to κ (automatically non-positive: $\kappa \leq 0$) yields equation (48). Similarly, equation (A24) of $\kappa\,\grave{R} = (\kappa\,-\,1)\,R$ turns to be an equation $\bar{\kappa}^{**}$ (≤ 0) in corollary 4. Next, substituting equations (48) and the equation $\bar{\kappa}^{**}$ back into equations (A21) and (A22), respectively, as

$$\text{(A25)}\quad q = \sqrt{\frac{-1}{2\,\kappa^{**} - 1}} = \sqrt{\frac{-1}{-2\,R/\{2\,(\grave{R} - R)\} - 1}} = \frac{\sqrt{(\grave{R} - R)\,\grave{R}}}{\grave{R}},$$

$$\text{(A26)}\quad q = \frac{1}{1 -\ \bar{\kappa}^{**}} = \frac{1}{1 + R/(\grave{R} - R)} = \frac{\grave{R} - R}{\grave{R}}.$$

By plugging substituted equations (A25) and (A26) back into equation (A17) and (A18), respectively, one has exactly the same equation (51). Finally, by substituting equations (A25) and (A26) into equation (4) with $x_r = \bar{x}_r^C\,q$ and with $x_r = \bar{x}_r^H\,q$ where $\bar{x}_r^C \equiv \sqrt{\gamma_r/a_r}$ and $\bar{x}_r^H \equiv \dfrac{\gamma_r}{b_r}$ as follows:

$$\text{(A27)}\quad \mu_r = \frac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \frac{\gamma_r/(\bar{x}_r^C\,q) - a_r\,\bar{x}_r^C\,q}{a_r\,\bar{x}_r^C\,q};$$

$$\text{(A28)}\quad \mu_r = \frac{\phi_r(x_r) - f_r(x_r)}{f_r(x_r)} = \frac{\gamma_r/(\bar{x}_r^H\,q) - b_r}{b_r},$$

one has nothing but equations (50) and (53), respectively. □

☞ POOF OF COROLLARY 5:

Proof. From equations (A12) and (A23), each market's tax revenue function of κ can be written as $4\,\hat{R}_1^{max}\,\dfrac{\kappa(\kappa-1)}{(2\,\kappa-1)^2}$, $\grave{R}_2^{max}\,\dfrac{2\kappa}{2\,\kappa-1}$, and $\grave{R}_3^{max}\,\dfrac{2\kappa}{2\,\kappa-1}$, in which $\hat{R}_1^{max} = \dfrac{(\beta_1 - b_1)^2}{4\,(\alpha_1 + a_1)}$, $\grave{R}_2^{max} = \gamma_2$, and $\grave{R}_3^{max} = \gamma_3$, respectively, as shown in equations (8) and (10). Those tax revenue functions of κ add up to equation (55), so that by setting it equal to a government's tax revenue R, we can have the following quadratic function of κ as

$$\text{(A29)}\quad 4\,(\hat{R}_{mix}^{max} - R)\,\kappa^2 - 2\,(\hat{R}_{mix}^{max} + \hat{R}_1^{max} - 2\,R)\,\kappa - R = 0$$

where $\hat{R}_{mix}^{max} \equiv \hat{R}_1^{max} + \grave{R}_2^{max} + \grave{R}_3^{max} = \dfrac{1}{4}\,\dfrac{(\beta_1 - b_1)^2}{\alpha_1 + a_1} + \gamma_2 + \gamma_3$. Due to a positive discriminant $\dfrac{D}{4} \equiv (\hat{R}_{mix}^{max} + \hat{R}_1^{max} - 2\,R)^2 + 4\,(\hat{R}_{mix}^{max} - R)\,R > 0$, in which $0 \leq R < \hat{R}_{mix}^{max}$, a negative real root in equation (A29) becomes equation (56) itself owing to the domain of $\kappa \leq 0$. □

B. *A Bordered Hessian*

In this appendix, we discuss the second-order condition for the Lagrange function $\mathcal{L}$ in equation (15) as well as a bordered Hessian $|\bar{H}|$ and its principal minors, $|\bar{H}_2|$, $|\bar{H}_3|$, $\cdots$, and $|\bar{H}_n|$ with the last one being that $|\bar{H}_n| = |\bar{H}|$.

Denote by g a sum of tax revenues R_r in equation (6) for $r = 1, 2, \cdots, n$ or $g \equiv \sum_{r=1}^{n} \lambda_r\, x_r$, and rewrite the Lagrange function $\mathcal{L}$ in equation (15) as

$$(15)\quad \underset{x_1, x_2, \cdots, x_n, \kappa}{\text{maximizes}}\ \mathcal{L} \equiv -\sum_{r=1}^{n} \int_{x_r}^{\bar{x}_r} \lambda_r\, ds_r + \kappa\,(R - g)$$

where κ is the multiplier, then we have a bordered Hessian $|\bar{H}|$ of

$$(\text{B1})\quad |\bar{H}| \equiv \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & \mathcal{L}_{11} & 0 & \cdots & 0 \\ g_2 & 0 & \mathcal{L}_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & 0 & 0 & \cdots & \mathcal{L}_{nn} \end{vmatrix}$$

including its successive bordered principal minors as

$$(\text{B2})\quad |\bar{H}_2| \equiv \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & 0 \\ g_2 & 0 & \mathcal{L}_{22} \end{vmatrix},\ |\bar{H}_3| \equiv \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & \mathcal{L}_{11} & 0 & 0 \\ g_2 & 0 & \mathcal{L}_{22} & 0 \\ g_3 & 0 & 0 & \mathcal{L}_{33} \end{vmatrix},\ \cdots,\ |\bar{H}_n|,$$

the last of which must be exactly the same as equation (B1) or $|\bar{H}_n| \equiv |\bar{H}|$ where all non-zero components of partial derivatives g_r and $\mathcal{L}_{rr}$ are supposed to be negative with $\frac{d\,R_r}{d\,x_r} \equiv R'_r < 0$ in equation (13) and with $\lambda'_r < 0$, $\kappa \leq 0$, and $\frac{d^2 R_r}{d\,x_r^2} \equiv R''_r \leq 0$ in equations (8), (A7), and (12), respectively, as

$$(\text{B3})\quad g_r \equiv \frac{\partial\, R_r}{\partial\, x_r} = \frac{d}{d\,x_r}(\lambda_r\, x_r) = R'_r = \lambda'_r\, x_r + \lambda_r < 0,$$

$$(\text{B4})\quad \mathcal{L}_{rr} \equiv \frac{\partial^2 \mathcal{L}}{\partial\, x_r^2} = \frac{d}{d\,x_r}\left(\frac{\partial\, \mathcal{L}}{\partial\, x_r}\right) = \frac{d}{d\,x_r}\{\lambda_r - \kappa\,(\lambda'_r\, x_r + \lambda_r)\}$$
$$= \lambda'_r - \kappa\,(\lambda''_r\, x_r + 2\,\lambda'_r) = \lambda'_r - \kappa\, R''_r < 0.$$

Therefore, principal minors in equation (B2) alternately become positive and negative as $|\bar{H}_2| = -\,\mathcal{L}_{22}\, g_1^2 - \mathcal{L}_{11}\, g_2^2 > 0$, $|\bar{H}_3| = \mathcal{L}_{33}\,|\bar{H}_2| - \mathcal{L}_{11}\,\mathcal{L}_{22}\, g_3^2 < 0$, $|\bar{H}_4| = \mathcal{L}_{44}\,|\bar{H}_3| - \mathcal{L}_{11}\,\mathcal{L}_{22}\,\mathcal{L}_{33}\, g_4^2 > 0$, $\cdots$, and so provided that $|\bar{H}_n| = (-1)^n |\bar{H}_n| > 0$ as Alpha C. Chiang (1984) @p.385, then we are always able to induce further in the Ramsey's tax problem like equation (15) here that $|\bar{H}_{n+1}| = \mathcal{L}_{n+1}\,|\bar{H}_n| - \mathcal{L}_{11}\,\mathcal{L}_{22} \cdots \mathcal{L}_{nn}\, g_{n+1}^2 < 0$ due to equations (B3) and (B4): $g_r < 0$; $\mathcal{L}_{rr} < 0$ for $r = 1, 2, \cdots, n$, and $n + 1$.

Then, all we have to do for the second-order condition corresponding to an optimal indirect tax is to check out whether or not all non-zero elements of the bordered Hessian $|\bar{H}|$ in equation (B1), or partial derivatives g_r and $\mathcal{L}_{rr}$ are negative: $g_r < 0$; $\mathcal{L}_{rr} < 0$ for $r = 1, 2, \cdots$, and n.

So, it is a piece of cake to see in the affine n-good markets of corollary 2 that every non-zero element of $g_r = R'_r$ or $\mathcal{L}_{rr} = \lambda'_r - \kappa\, R''_r$ is negative as follows: ✍❶ Because equations (40) and (41) give us the vertical length λ_r in equation (1) or (3) as $\lambda_r \equiv -(\alpha_r + a_r)\, x_r + \beta_r - b_r$ in the domain of $0 < x_r \leq \bar{x}_r^B \equiv (\beta_r - b_r)/(\alpha_r + a_r)$ from equation (2), its derivative λ'_r and the second one λ''_r are calculated as $\lambda'_r = -(\alpha_r + a_r) < 0$ and $\lambda''_r = 0$, respectively. ✍❷ Those equations also give us the tax revenue of R_r in equation (6) as $R_r = \lambda_r\, x_r = -(\alpha_r + a_r)\, x_r^2 + (\beta_r - b_r)\, x_r$ so that its derivative of $R'_r = \lambda'_r\, x_r + \lambda_r$ and the second one of $R''_r = \lambda''_r\, x_r + 2\,\lambda'_r$ can be obtained as $R'_r = -2\,(\alpha_r + a_r)\, x_r + \beta_r - b_r < 0$ as long as $0.5\,\bar{x}_r^B < x_r \leq \bar{x}_r^B$ or $0.5 < q = q_r \equiv x_r/\bar{x}_r^B \leq 1$ in terms of the unit-less quantity q in equation (A5), but as $R''_r = 2\,\lambda'_r < 0$ all the time. ✍❸ Recall in equation (A10) that the multiplier κ keeps its domain of $\kappa \leq 0$ as long as $0.5 < q = q_r \equiv x_r/\bar{x}_r^B \leq 1$, and so that let x_r^* be an equilibrium quantity corresponding to an optimal indirect tax or $0.5 < q = q_r \equiv x_r^*/\bar{x}_r^B \leq 1$, then we always have $g_r = R'_r < 0$, $\kappa \leq 0$, and $\mathcal{L}_{rr} = (1 - 2\,\kappa)\,\lambda'_r < 0$ for all $r = 1, 2, \cdots, n$. In fact, we have shown an optimal κ as $\kappa^* = \{\hat{R} - R - \sqrt{(\hat{R} - R)\,\hat{R}}\,\}/\{2\,(\hat{R} - R)\} \leq 0$ in equation (43).

Similarly, in the n-good markets of corollary 3, ✍❹ since equations (46) and (47) yield the vertical length of λ_r in equation (1) or (3) as $\lambda_r \equiv \gamma_r/x_r - a_r\, x_r$ in the domain of $0 < x_r \leq \bar{x}_r^C \equiv \sqrt{\gamma_r/a_r}$ from equation (2), its derivative λ'_r and the second one λ''_r are computed as $\lambda'_r = -\gamma_r/x_r^2 - a_r < 0$ and $\lambda''_r = 2\,\gamma_r/x_r^3 > 0$. ✍❺ The tax revenue of R_r in equation (6) is given as $R_r = \lambda_r\, x_r = \gamma_r - a_r\, x_r^2$ so that its derivative R'_r and the second one R''_r can be computed as $R'_r = -2\, a_r\, x_r < 0$ and $R''_r = \lambda''_r\, x_r + 2\,\lambda'_r = -2\, a_r < 0$. ✍❻ Recall in equation (A19) that κ has a domain of $\kappa \leq 0$ with a region of the unit-less q as $0 < q = q_r \equiv x_r^{**}/\bar{x}_r^C \leq 1$ for an optimal equilibrium quantity after tax x_r^{**}, then one always has $g_r = R'_r < 0$, $\kappa \leq 0$, and $\mathcal{L}_{rr} = \lambda'_r - \kappa\, R''_r = \lambda'_r + 2\, a_r\, \kappa < 0$ for all $r = 1, 2, \cdots$, and n. Actually, she or he has had an optimal κ as $\kappa^{**} = -R/\{2\,(\grave{R} - R)\} \leq 0$ in equation (48).

Moreover, in the n-good markets of corollary 4, ✍❼ since equations (46) and (52) give us the vertical length λ_r in equation (1) or (3) as $\lambda_r \equiv \gamma_r/x_r - b_r$ in the domain of $0 < x_r \leq \bar{x}_r^H \equiv \gamma_r/b_r$ from equation (2), its derivative λ'_r

and the second one λ_r'' are computed as $\lambda_r' = -\gamma_r/x_r^2 < 0$ and $\lambda_r'' = 2\,\gamma_r/x_r^3 > 0$. ✍❽ The tax revenue of R_r in equation (6) is given as $R_r = \lambda_r\, x_r = \gamma_r - b_r\, x_r$ so that its derivative R_r' and the second one R_r'' can be calculated as $R_r' = -\,b_r < 0$ and $R_r'' = \lambda_r''\, x_r + 2\,\lambda_r' = 0$. ✍❾ Recall in equation (A20) that κ has a domain of $\kappa \le 0$ with a region of the unit-less q as $0 < q = q_r \equiv \bar{x}_r^{**}/\bar{x}_r^{H} \le 1$ for an optimal equilibrium quantity after tax $\bar{x}_r^{**}$, then we always have that $g_r = R_r' < 0$, $\kappa \le 0$, and that $\mathcal{L}_{rr} = \lambda_r' - \kappa\, R_r'' = \lambda_r' < 0$ for all $r = 1,\, 2,\, \cdots$, and n, in which we have already had a non-positive optimal κ as $\bar{\kappa}^{**} = -R/(\grave{R} - R) \le 0$ in corollary 4.

Furthermore, it can be observed in corollary 5 with parameters $a_3 = \dfrac{1}{3}$, $\alpha_1 = \gamma_2 = a_1 = b_1 = 1$, $\gamma_3 = 3$, $a_2 = 4$, and $\beta_1 = 5$ reflecting four optimal choice variables, the multiplier $\tilde{\kappa}_{mix}^{*} = -0.5$ in equation (57), equilibrium quantities after tax $\tilde{x}_1^{*} = 1.5$, $\tilde{x}_2^{*} = \sqrt{2}/4$, and $\tilde{x}_3^{*} = 3\,\sqrt{2}/2$ that all non-zero elements in a bordered Hessian are negative as follows: ✌① As shown in ✍❶ and ✍❹, the derivatives λ_r' are computed as $\lambda_1' = -(\alpha_1 + a_1) = -2 < 0$, $\lambda_2' = -\gamma_2/(\tilde{x}_2^{*})^2 - a_2 = -12 < 0$, and $\lambda_3' = -\gamma_3/(\tilde{x}_3^{*})^2 - a_3 = -1 < 0$, respectively. ✌② Besides, as shown in ✍❷ and ✍❺, the derivatives R_r' are calculated as $R_1' = -2\,(\alpha_1 + a_1)\,\tilde{x}_1^{*} + \beta_1 - b_1 = -2 < 0$, $R_2' = -2\,a_2\,\tilde{x}_2^{*} = -2\sqrt{2} < 0$, and $R_3' = -2\,a_3\,\tilde{x}_3^{*} = -\sqrt{2} < 0$, respectively; on the other hand, the second derivatives R_r'' are obtained as $R_1'' = -2\,(\alpha_1 + a_1) = -4 < 0$, $R_2'' = -2\,a_2 = -8 < 0$, and $R_3'' = -2\,a_3 = -2/3 < 0$, respectively. ✌③ As seen in ✍❸ and ✍❻, therefore, all non-zero elements $g_r = R_r'$ as well as $\mathcal{L}_{rr} = \lambda_r' - \tilde{\kappa}_{mix}^{*}\, R_r''$ are strictly negative as $g_1 = -2 < 0$, $g_2 = -2\sqrt{2} < 0$, $g_3 = -\sqrt{2} < 0$, $\mathcal{L}_{11} = -4 < 0$, $\mathcal{L}_{22} = -16 < 0$, and $\mathcal{L}_{33} = -4/3 < 0$, respectively.

C. *Equations (3) and (11) in Frank P. Ramsey*

In this appendix, we review equations (3) and (11) (where we underline to let them different from ours) in Frank P. Ramsey (1927). To begin with, recall in equation (6) and in figure 1 that the government's tax revenue of

$$(6) \quad R_r \equiv \lambda_r\, x_r = \{\phi_r(x_r) - f_r(x_r)\}\, x_r$$

is geometrically described by a rectangular area or a product of the vertical length λ_r in equation (1) or (3) times the width of the quantity x_r for $r = 1$, $2, \cdots, n$. In each market, according to equations (7) and (9), the maximum tax revenue R_r^{max} of equation (6) ought to retain equation (11) or

$$\text{(C1)} \quad \frac{d\,R_r}{d\,x_r} \equiv \lambda_r'\, x_r + \lambda_r \le 0$$

at $x_r = x_r^{max}$; this equal sign in equation (C1) or equation (7) gives us

$$\text{(C2)}\quad \frac{-\lambda_r}{\lambda_r'\, x_r} = 1.$$

Now, denote by θ_r an elasticity of width x_r with respect to the length λ_r, *i.e.*, the length elasticity of width in the geometrical rectangle reflecting the tax revenue R_r in equation (6) as shown in figure 1 as

$$\text{(C3)}\quad \theta_r \equiv \frac{-\lambda_r}{\lambda_r'\, x_r} = \frac{-\{\phi_r(x_r) - f_r(x_r)\}}{\{\phi_r'(x_r) - f_r'(x_r)\}\, x_r},$$

then equations (C1), (C2), (5), and (8) yield the following regions to us:

$$\text{(C4)}\quad \frac{d\,R_r}{d\,\mu_r} = \lambda_r'\, x_r(1-\theta_r)\,\frac{1}{\mu_r'} = \begin{cases} \geq 0 & \text{if } 0 \leq \theta_r < 1; \\ = 0 & \text{if } \theta_r = 1 \text{ with } R_r^{max} = \hat{R}_r^{max}; \\ < 0 & \text{if } \theta_r > 1; \end{cases}$$

owing to the chain rule of equations (C1) and (5):

$$\frac{d\,R_r}{d\,\mu_r} = \frac{d\,R_r}{d\,x_r}\,\frac{d\,x_r}{d\,\mu_r} = (\lambda_r'\, x_r + \lambda_r)\,\frac{d\,x_r}{d\,\mu_r} = \lambda_r'\, x_r(1 - \frac{-\lambda_r}{\lambda_r'\, x_r})\,\frac{1}{\mu_r'}.$$

As discussed in footnote 7, equation (3) in Frank P. Ramsey (1927) @p.56 is our equation (C3), but it should have been identical to

$$\text{(C5)}\quad \frac{\partial\, u_r/\partial\, x_r}{\partial\, R_r/\partial\, x_r} = \frac{\lambda_r}{\lambda_r + \lambda_r'\, x_r} = -K$$

for $r = 1, 2, \cdots, n$. So, it is easy to show that the multiplier K turns to be

$$\text{(C6)}\quad K = -\frac{\lambda_r/(\lambda_r'\, x_r)}{\lambda_r/(\lambda_r'\, x_r) + 1} = \frac{\theta_r}{1-\theta_r}$$

in terms of the elasticity $\theta_r \equiv \dfrac{-\lambda_r}{\lambda_r'\, x_r}$ in equation (C3). Since equation (C6) holds for all r and $s \in \{r \mid r = 1, 2, \cdots, n\}$, one might ignore or eliminate the multiplier K and set $\dfrac{\theta_r}{1-\theta_r} = \dfrac{\theta_s}{1-\theta_s}$ to obtain $\theta_r\,(1-\theta_s) = \theta_s\,(1-\theta_r)$ and redundantly take $\theta = \theta_r = \theta_s$ as a kind of new multiplier like

$$\text{(C7)}\quad K = \frac{\theta}{1-\theta} = \begin{cases} \geq 0 & \text{if } 0 \leq \theta < 1, \\ = \pm\infty & \text{if } \theta = 1 \text{ with } R_r^{max} = \hat{R}_r^{max}, \\ < 0 & \text{if } \theta > 1, \end{cases}$$

in which the multiplier K has a jump region at $\theta = 1$. It is easy to calculate its derivative with respect to θ or $\dfrac{d\,K}{d\,\theta} = \dfrac{1}{(1-\theta)^2} > 0$ everywhere at $\theta \neq 1$.

Next, it can be shown from equations (1) or (3), (4), (C3), and (C7) that

$$\text{(C8)}\quad \mu_r = \frac{\lambda_r}{f_r} = \frac{-\lambda_r}{\lambda_r'\, x_r}\,\frac{-\lambda_r'\, x_r}{f_r} = \theta_r\,\frac{-\lambda_r'\, x_r}{f_r} = \theta\,\frac{-\lambda_r'\, x_r}{f_r},$$

and that equation (C8) becomes equation (11) in Frank P. Ramsey (1927) as

$$\text{(C9)}\quad \mu_r = \frac{\theta(1/\varepsilon_r + 1/\rho_r)}{1-\theta/\rho_r}$$

in terms of three elasticities: ρ_r and ε_r as well as θ; *i. e.*, two price elasticities of demand $\rho_r \equiv -\dfrac{\phi_r}{\phi_r'\, x_r}$ and supply $\varepsilon_r \equiv \dfrac{f_r}{f_r'\, x_r}$ in equation (29) as well as the length elasticity of width $\theta = \theta_r \equiv \dfrac{-\lambda_r}{\lambda_r'\, x_r}$ in equation (C3), respectively, because we are able to put equation (C8) into $\mu_r = \theta\ (f_r'\, x_r/f_r\ -\ \phi_r'\, x_r/f_r)$ $= \theta\ [f_r'\, x_r/f_r\ -\ \phi_r'\, x_r/\{\phi_r/(1\ +\ \mu_r)\}]\ =\ \theta\ [1/(f_r/f_r'\, x_r)\ +\ (1\ +\ \mu_r)/\{-\phi_r/(\phi_r'\, x_r)\}] = \theta\ \{1/\varepsilon_r\ +\ (1 + \mu_r)\ (1/\rho_r)\} = \theta\ (1/\varepsilon_r\ +\ 1/\rho_r)\ +\ \mu_r\,\theta/\rho_r$ due to $\phi_r\ =\ (1\ +\ \mu_r)\, f_r$ from equation (3).

So, other things being equal in equation (C9), Frank P. Ramsey (1927) unusually treats only θ as a running parameter in taking its limit as θ approaches zero from the right (0^+) so as to claim a proportionality ($\propto$) of

$$\text{(C10)}\quad \mu_r \propto \left(\frac{1}{\varepsilon_r} + \frac{1}{\rho_r}\right) \equiv \sigma_r$$

although his unusual limit does not seem to result in equation (C10) but in

$$\text{(C11)}\quad \mu_r = \frac{\theta(1/\varepsilon_r + 1/\rho_r)}{1-\theta/\rho_r} \to 0\ \text{ as } \theta \to 0^+.$$

Therefore, it is quite apparent as proved in proposition 1 such an infinitesimal ad valorem tax rate μ_r could not have been proportional to the sum function σ_r in equation (29) at all as

$$\text{(C12)}\quad \mu_r \not\propto \left(\frac{1}{\varepsilon_r} + \frac{1}{\rho_r}\right) \equiv \sigma_r$$

everywhere in the domain of equation (2) or $0 < x_r \leq \bar{x}_r$. It is also quite apparent from equation (C9) that the ad valorem tax rate $\mu_r \propto \theta$, the length elasticity of width.

Equation (C9) just provides us with another $\dfrac{0}{0}$ form, a part of which is well known as the l'Hôpital's rule such as

$$\text{(l'Hôpital)}\quad \frac{0}{0} = \lim_{x_r \to \bar{x}_r^-} \frac{\mu_r}{\theta} = \lim_{x_r \to \bar{x}_r^-} \frac{\{\mu_r(x_r) - \mu_r(\bar{x}_r)\}/(x_r - \bar{x}_r)}{\{\theta(x_r) - \theta_r(\bar{x}_r)\}/(x_r - \bar{x}_r)} = \frac{\mu_r'(\bar{x}_r)}{\theta'(\bar{x}_r)}$$

because of $\mu_r(\bar{x}_r) = \theta(\bar{x}_r) = 0$ from equations (4) and (C3), and so that

$$\text{(C13)}\quad \frac{\mu_r'(\bar{x}_r)}{\theta'(\bar{x}_r)} = \frac{d\,\mu_r/d\,x_r}{d\,\theta/d\,x_r}\bigg|_{x_r=\bar{x}_r} = \frac{d\,\mu_r}{d\,\theta}\bigg|_{x_r=\bar{x}_r} = \frac{d\,\mu_r}{d\,\theta}\bigg|_{\mu_r=\theta_r=0}.$$

In fact, the $\frac{0}{0}$ form of equation (l'Hôpital) or (C13) becomes

$$\text{(C14)}\quad \frac{0}{0} = \lim_{x_r\to\bar{x}_r^-}\frac{\mu_r}{\theta} = \lim_{x_r\to\bar{x}_r^-}\frac{\lambda_r(x_r)/f_r(x_r)}{-\lambda_r(x_r)/\{\lambda_r'(x_r)\,x_r\}} = -\lambda_r'(\bar{x}_r)\frac{\bar{x}_r}{f(\bar{x}_r)},$$

then the fact of $f_r(\bar{x}_r) = \phi_r(\bar{x}_r)$ at $x_r = \bar{x}_r$ in equation (1) or (3) brings us

$$\begin{aligned}-\lambda_r'(\bar{x}_r)\frac{\bar{x}_r}{f(\bar{x}_r)} &= \frac{f_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} - \frac{\phi_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)}\\ &= \frac{f_r'(\bar{x}_r)\,\bar{x}_r}{f_r(\bar{x}_r)} - \frac{\phi_r'(\bar{x}_r)\,\bar{x}_r}{\phi_r(\bar{x}_r)} = \sigma_r(\bar{x}_r) > 0.\end{aligned}$$

As discussed in equation (A4) that the term $[\{\lambda_r(x_r)\}]$ should be null even for an infinitesimal rate μ_r, it seems worth reminding here that in a denominator of equation (C14), one needs $[\{\lambda_r(x_r)\}] = \phi_r(x_r) - f_r(x_r) \approx 0$ at $x_r < \bar{x}_r$ as a proxy for the fact of $f_r(\bar{x}_r) = \phi_r(\bar{x}_r)$ at $x_r = \bar{x}_r$ in equation (1) or (3).

At last, hence, the limit of the $\frac{0}{0}$ form from equation (C9) gives us exactly the same result from proposition 1 or equations (A3) and (A4) as well as one from the above equation (C14) as follows:

$$\text{(C15)}\quad \frac{0}{0} = \lim_{x_r\to\bar{x}_r^-}\frac{\mu_r}{\theta} = \lim_{x_r\to\bar{x}_r^-}\frac{\frac{1}{\varepsilon_r}+\frac{1}{\rho_r}}{1-\frac{\theta}{\rho_r}} = \frac{\lim_{x_r\to\bar{x}_r^-}\sigma_r}{1-0} = \sigma_r(\bar{x}_r) > 0,$$

which immediately tells us that equation (11) in Frank P. Ramsey (1927) or equation (C9) cannot produce a fruit of the proportionality at all on the sum of reciprocal price elasticities like $\mu_r \not\propto \frac{1}{\varepsilon_r} + \frac{1}{\rho_r}$ although equations (C15), (22) through (26) can produce it on the length elasticity of width θ in equation (C3) like $\mu_r \propto \theta$, on the tax revenue R_r like $\mu_r \propto R_r$ in equation (27), on the percentage-reduction K_r like $\mu_r \propto K_r$ in equation (A4), and so forth.

In terms of the *Infinitesimal Calculus* as Keith D. Stroyan (1993) @p.37-42, by the way, it can be seen that a quotient limit from equation (11) in Frank P. Ramsey (1927) or our equation (C9) diverges as

$$\text{(C16)}\quad \lim_{x_r\to\bar{x}_r^-}\frac{\frac{1}{\varepsilon_r}+\frac{1}{\rho_r}}{\mu_r} = \lim_{x_r\to\bar{x}_r^-}\frac{1-\frac{\theta_r}{\rho_r}}{\theta_r} = \frac{1+\lim_{x_r\to\bar{x}_r^-}\frac{\lambda_r\,\phi_r'}{\lambda_r'\phi_r}}{\lim_{x_r\to\bar{x}_r^-}\frac{-\lambda_r}{\lambda_r'\,x}} \equiv \frac{1+\zeta_{\lambda_r}^+}{\zeta_{\theta_r}^+} \to +\infty$$

where we treat the length λ_r in equation (1) or (3) as an infinitesimal from the right denoted by $\zeta^+_{\lambda_r}$ such that $0 < \zeta^+_{\lambda_r} \approx 0$ as well as its corresponding length elasticity θ_r in equation (C3) as an infinitesimal from the right of $\zeta^+_{\theta_r}$ such that $0 < \zeta^+_{\theta_r} \approx 0$ due to equation (23) as $\lim_{x_r \to \bar{x}_r^-} \lambda_r = \phi_r(\bar{x}_r) - f_r(\bar{x}_r) = 0$ and to the limit of equation (C3) as $\lim_{x_r \to \bar{x}_r^-} \theta_r = \dfrac{-\{\phi_r(\bar{x}_r) - f_r(\bar{x}_r)\}}{\{\phi'_r(\bar{x}_r) - f'_r(\bar{x}_r)\}\,\bar{x}_r} = 0$. As proved in proposition 1, therefore, the sum function $\sigma_r = \dfrac{1}{\varepsilon_r} + \dfrac{1}{\rho_r}$ in equation (29) is always too large as infinity in equation (C16) to have the same order as the limit of equation (C3) and equations (22) through (26).

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